Neumaier graphs

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(joint work with A. Abiad, W. Castryck,
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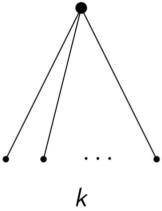
Combinatorial Designs and Codes – Rijka

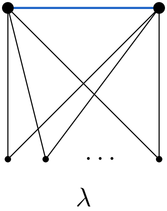
Regular graphs

Regularity in graphs

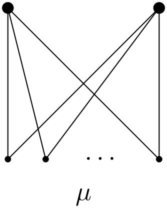
k-regular



edge-regular



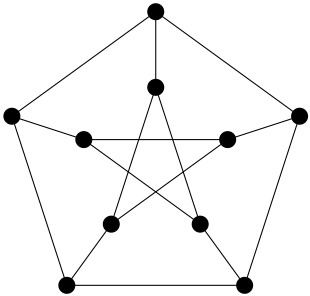
co-edge-regular



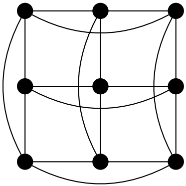
3 Strongly regular graphs

Definition

A regular graph is ***strongly regular*** if it is edge-regular and co-edge-regular.



The Petersen graph
srg$(10, 3, 0, 1)$



The 3×3 rook's graph
srg$(9, 4, 1, 2)$

4 Regularity of subsets

Definition

A vertex subset S is *e-regular* if for every vertex $x \notin S$ we have $|N(x) \cap S| = e$.

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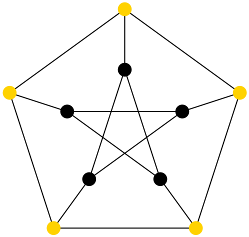
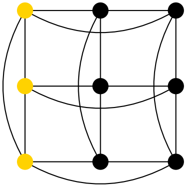


1-regular subset
No regular cliques



A 1-regular clique

5 Neumaier's question

Theorem (Neumaier, 1981)

A vertex-transitive and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier)

Is a regular, edge-regular graph with a regular clique necessarily strongly regular?

5 Neumaier's question

Theorem (Neumaier, 1981)

A vertex-transitive and edge-transitive graph with a regular clique is strongly regular.

Problem (Neumaier)

Is a regular, edge-regular graph with a regular clique necessarily strongly regular?

Definition

A ***Neumaier graph*** is a regular, edge-regular graph with a regular clique. It is a ***strictly Neumaier graph*** if it is not strongly regular.
A Neumaier graph has parameters $(v, k, \lambda; e, s)$ if it is an edge-regular graph with parameters (v, k, λ), admitting an e-regular clique of size s.

Theorem (folklore; Neumaier; Evans-Goryainov-Panasenko)

If there is a Neumaier graph with parameters $(v, k, \lambda; e, s)$*, then*

(i) $v > k > \lambda$ *and* $v - 2k + \lambda \geq 0$*;*

(ii) $vk \equiv 0 \pmod 2$*,* $k\lambda \equiv 0 \pmod 2$ *and* $vk\lambda \equiv 0 \pmod 6$*;*

6 Counting

Theorem (folklore; Neumaier; Evans-Goryainov-Panasenko)

If there is a Neumaier graph with parameters $(v, k, \lambda; e, s)$, *then*

(i) $v > k > \lambda$ *and* $v - 2k + \lambda \geq 0$;

(ii) $vk \equiv 0 \pmod 2$, $k\lambda \equiv 0 \pmod 2$ *and* $vk\lambda \equiv 0 \pmod 6$;

(iii) $s(k - s + 1) = (v - s)e$;

(iv) $s(s - 1)(\lambda - s + 2) = (v - s)e(e - 1)$;

(v) $k - s + e - \lambda - 1 \geq 0$.

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(v) $k - s + e - \lambda - 1 \geq 0$.

If there is a strictly Neumaier graph with parameters $(v, k, \lambda; e, s)$, *then moreover*

(i*) $v - 1 > k$ *and* $v - 2k + \lambda \geq 2$;

(v*) $k - s + e - \lambda - 1 \geq 1$;

(vi) $\lambda + 3 > s \geq 4$;

(vii) $1 \leq e < s - 1$

7 More counting!

Theorem (Abiad-DB-Koolen-Zeijlemaker, 2021)

If there is a Neumaier graph with parameters $(v, k, \lambda; e, s)$*, then* $(v-k-1)(v-k-2) - k(v-2k+\lambda) \geq 0$.
If there is a strictly Neumaier graph with parameters $(v, k, \lambda; e, s)$*, then* $(v-k-1)(v-k-2) - k(v-2k+\lambda) > 0$.

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If there is a strictly Neumaier graph with parameters $(v, k, \lambda; e, s)$, *then* $(v-k-1)(v-k-2) - k(v-2k+\lambda) > 0$.

Theorem (Abiad-DB-Koolen-Zeijlemaker, 2021)

There is no (strictly) Neumaier graph with parameter set $(6l+3, 4l+2, 3l; l+1, 2l+1)$ *for any integer* $l \geq 3$.

8 More counting: the consequences

Example

These are no longer admissible parameter sets for Neumaier graphs:

$$\left(3\frac{a^{n+1}-1}{a-1}, 2a^n + a\frac{a^n-1}{a-1}, 3a^n - 2a^{n-1} + a\frac{a^{n-1}-1}{a-1} - 1; a^n, \frac{a^{n+1}-1}{a-1}\right)$$

$$(27b+21, 21b+14, 13b+7; 6b+4, 9b+7) \qquad a, n, b \in \mathbb{N}, a \geq 3, n \geq 2, b \geq 1.$$

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$$(27b+21, 21b+14, 13b+7; 6b+4, 9b+7) \qquad a, n, b \in \mathbb{N}, a \geq 3, n \geq 2, b \geq 1.$$

These are no longer admissible parameter sets for strictly Neumaier graphs:

$$(2(2a+1)(a^2+a-1), 2(a+1)(2a^2-1), 4a^3+2a^2+a-3; 4a^2-2a, 4a^2-1)$$

$$(a^2(2a+3), (a+1)(4a^2-1), 4a^3+2a^2+a-2; 4a^2-2a, 4a^2) \qquad a \in \mathbb{N}, a \geq 2.$$

9 A first look at the table

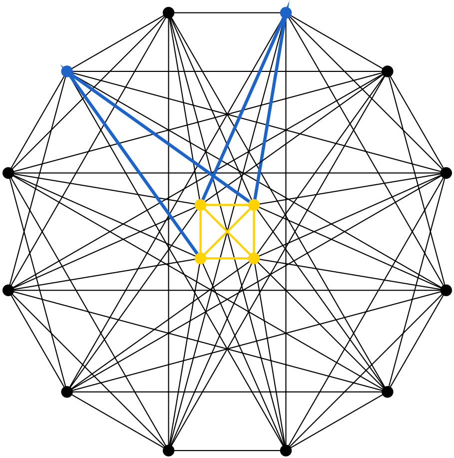
v	k	λ	e	s
16	9	4	2	4
21	14	9	4	7
22	12	5	2	4
24	8	2	1	4
25	12	5	2	5
	16	9	3	5
26	15	8	3	6
27	18	12	5	9
28	9	2	1	4
	15	6	2	4
		8	3	7
	18	11	4	7
33	22	15	6	11
	24	17	6	9

v	k	λ	e	s
34	18	7	2	4
35	10	3	1	5
	16	6	2	5
	18	9	3	7
	22	12	3	5
36	11	2	1	4
	15	6	2	6
	20	10	3	6
	21	12	4	8
	25	16	4	6
39	26	18	7	13
	30	23	9	13

The previous theorems take care of 13 of the 99 cases with $v \leq 64$.

Existence

10 Strictly Neumaier graphs exist



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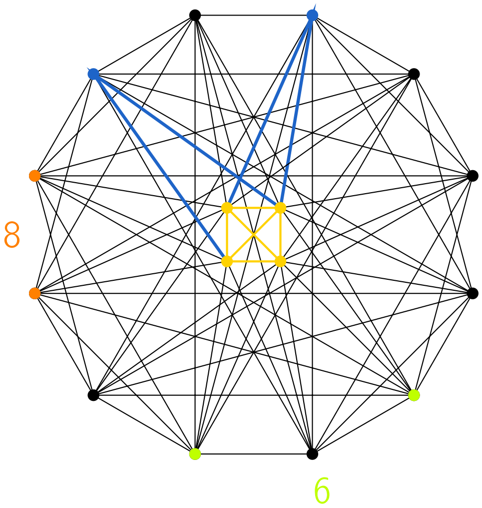

And there is an infinite number of them

Theorem (Greaves-Koolen, 2017)

There are strictly Neumaier graphs.

And there is an infinite number of them

Theorem (Greaves-Koolen, 2017)

There are strictly Neumaier graphs.

Theorem (Evans-Goryainov-Panasenko)

There is an infinite class of strictly Neumaier graphs with parameters $(2^{2n}, (2^{n-1}+1)(2^n-1), 2(2^{n-2}+1)(2^{n-1}-1); 2^{n-1}, 2^n)$

12 Evans' machine

Inspired by the Greaves-Koolen construction.

Theorem (Evans)

Let $\Gamma_1 = (V_1, E_1), \ldots, \Gamma_t = (V_t, E_t)$ be t edge-regular graphs with parameters (v, k, λ) such that each Γ_i admits a spread of 1-regular cocliques, $C_{i,1}, \ldots, C_{i,k+1}$. The graph $F(\Gamma_1, \ldots, \Gamma_t)$ is the graph

- *with as vertex set $V_1 \cup \cdots \cup V_t$,*
- *and where two vertices $x \in C_{i,k}$ and $y \in C_{j,l}$ are adjacent if and only if $i = j$ and $x \sim y$ in Γ_i, or if $k = l$.*

If $t = \frac{(\lambda+2)(k+1)}{v} \in \mathbb{N}$, then $F(\Gamma_1, \ldots, \Gamma_t)$ is a Neumaier graph with parameters $(vt, k + \lambda + 1, \lambda; 1, \lambda + 2)$; it admits a spread of 1-regular cliques.

Spread: family of pairwise disjoint subsets whose union is the whole vertex set.

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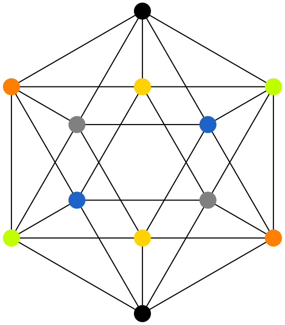
If $t \geq 2$, then $F(\Gamma_1, \ldots, \Gamma_t)$ is a strictly Neumaier graph.

Spread: family of pairwise disjoint subsets whose union is the whole vertex set.

13 A strictly Neumaier graph on 24 vertices



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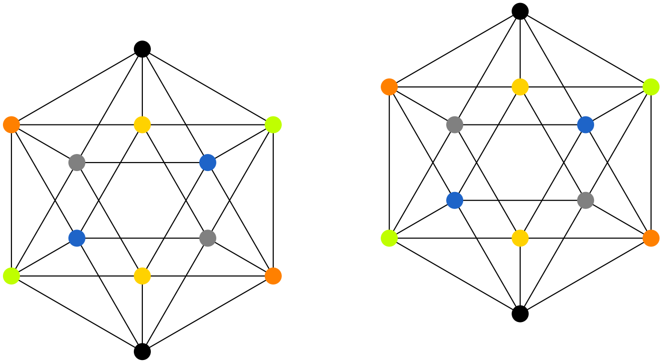

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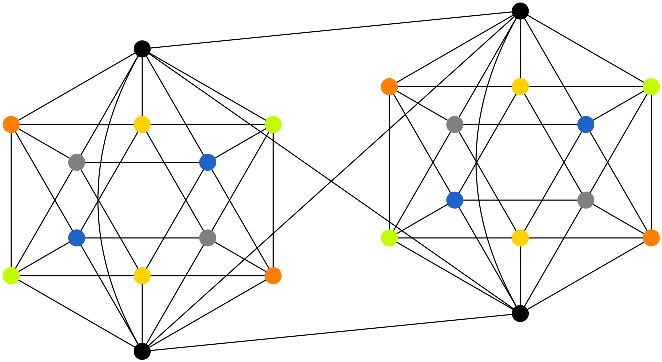

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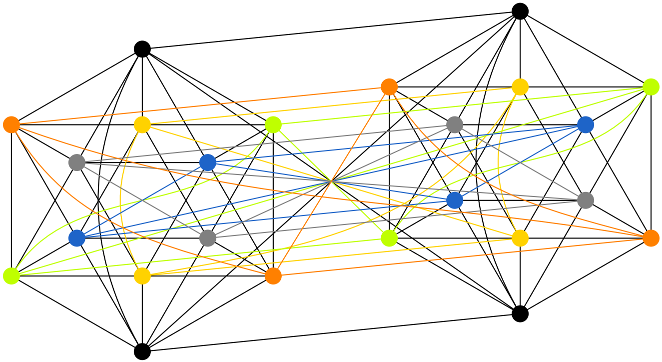

What to feed to the machine?

Theorem (Greaves-Koolen)

Take $V_1, \ldots, V_t$ distance-regular a-antipodal graphs of diameter 3.

Example

- Taylor graphs
- Thas-Somma graphs, edge-regular graphs with parameters $(q^{2n+1}, q^{2n}-1, q^{2n-1}-2)$ for a prime power q. You need to take q^{2n-2} copies. You get a strictly Neumaier graph with parameters $(q^{4n-1}, q^{2n-1}(q+1)-2, q^{2n-1}-2; 1, q^{2n-1})$.

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Theorem (Greaves-Koolen)

Take $V_1, \ldots, V_t$ a (specificly described) Cayley graph on $(\mathbb{Z}/2\mathbb{Z})^m \times (\mathbb{F}_q, +)$, with $m \in \{2, 3\}$ and q a prime power with $q \equiv 1 \pmod{2^{m+1} - 2}$.

$m = 2$: $q \in \{7, 13, 19, 37, 49, \ldots\}$, $m = 3$: $q \in \{29, 43, 71, 127, \ldots\}$

A second look at the table

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A new construction

17 Example

A strictly Neumaier graph on 65 vertices was known ... to which family does it belong?

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- $S_{65} = \{1, 2, 4, 8, 16, 32, 64 = -1, 63, 61, 57, 49, 33\}$

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- $p = 13$, $q = 5$, $a = 2$
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- $\Gamma_{65}(2)$ is edge-regular with parameters $(65, 12, 3)$, and has a spread of 1-regular cocliques: cosets of $\{0, 13, 26, 39, 52\}$ in $\mathbb{Z}/65\mathbb{Z}, +$

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- $t = \frac{(\lambda+2)(k+1)}{v} = \frac{(3+2)(12+1)}{65} = 1$
- $F(\Gamma_{65}(2))$ is a strictly Neumaier graph.

Theoretically

Definition

Let a be such that $a^i \equiv -1 \pmod{n}$, where $2i$ is the order of a in $(\mathbb{Z}/n\mathbb{Z})^*, \cdot$.
Then $S_n(a) = \{a^j \in \mathbb{Z}/n\mathbb{Z} \mid 0 \leq j < 2i\}$.
$\Gamma_n(a)$ is the Cayley graph on $\mathbb{Z}/n\mathbb{Z}, +$ with $S_n(a)$ as generating set.

Theorem (Abiad-DB-Koolen-Zeijlemaker)

Let $p > 2$ be a prime, $q \in \mathbb{N}$ odd. Let $a \in \mathbb{Z}$ be such that has order $p-1$ in $(\mathbb{Z}/p\mathbb{Z})^*, \cdot$ and such that $a^{\frac{p-1}{2}} \equiv -1 \pmod{pq}$.
Then, the Cayley graph $\Gamma_{pq}(a)$ is an edge-regular graph with parameters $(pq, p-1, \lambda)$, with $\lambda = |S_{pq}(a) \cap (S_{pq}(a) + 1)|$, that has a spread of 1-regular cocliques.

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Remark

In general we need that $\frac{(\lambda+2)(k+1)}{v} = \frac{|S_{pq}(a) \cap (S_{pq}(a)+1)|+2}{q}$ is an integer. In other words, $|S_{pq}(a) \cap (S_{pq}(a) + 1)| \equiv -2 \pmod{q}$.

19 Overview of new examples

q	p	a	t	v	k	λ	s
5	13	2	1	65	16	3	5
	37	2	1	185	40	3	5
	61	17	4	1220	79	18	20
	149	13	4	2980	167	18	20
		2	7	5215	182	33	35
7	79	54	1	553	84	5	7
	103	45	1	721	108	5	7
	127	12	2	1778	139	12	14
	139	10	2	1946	151	12	14
		26	4	3892	165	26	28
11	131	2	1	1441	140	9	11
	431	94	4	18964	473	42	44
13	61	2	1	793	72	11	13
	277	15	2	7202	301	24	26
	397	6	2	10322	421	24	26
		20	2	10322	421	24	26

19 Overview of new examples

q	p	a	t	v	k	λ	s
25	1021	77	2	51050	1069	48	50
		122	2	51050	1069	48	50
	1181	42	2	59050	1229	48	50
	1301	3	2	65050	1349	48	50
		73	2	65050	1349	48	50
	1381	42	2	69050	1429	48	50
		123	2	69050	1429	48	50
	1621	88	2	81050	1669	48	50
		113	2	81050	1669	48	50
	1741	197	2	87050	1789	48	50
	2141	58	2	107050	2189	48	50
		112	2	107050	2189	48	50
49	1303	754	2	127694	1399	96	98
	2311	129	2	226478	2407	96	98

Some number theory in the background

Problem

For which q can we find primes p such that the construction produces a strictly Neumaier graph? Does it produce an infinite number of examples?
We need to look at $|S_{pq}(a) \cap (S_{pq}(a)+1)| \pmod{q}$. Is it -2?

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Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker)

For $q = \prod_{i=1}^{m} p_i^{n_i}$ we have $|S_{pq}(a) \cap (S_{pq}(a)+1)| = 0$ if there is an i such that $n_i = 1$ and $p_i \geq 5$ is a Fermat prime, and there is a j such that $p_j \equiv 3 \pmod{4}$.

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Theorem (Abiad-Castryck-DB-Koolen-Zeijlemaker)

For any q and p, and any a fulfilling the conditions we have $|S_{pq}(a) \cap (S_{pq}(a)+1)| \equiv 0, 2 \pmod 3$.

Corollary (Abiad-Castryck-DB-Koolen-Zeijlemaker)

Our construction produces no new examples of (strictly) Neumaier graphs if $3|q$.

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Corollary (Abiad-Castryck-DB-Koolen-Zeijlemaker)

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Remark

Numerical evidence suggests in all other cases there are examples (probably an infinite number of them).

Eigenvalues

21 The number of eigenvalues

Theorem

Strongly regular graphs have three eigenvalues, one of them the parameter of regularity k (with multiplicity 1).

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Theorem (Abiad-De Bruyn-D'haeseleer-Koolen)

A strictly Neumaier graph cannot have four eigenvalues.

Example (Simoens)

There are many strictly Neumaier graphs with six eigenvalues.

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There are many strictly Neumaier graphs with six eigenvalues.

Theorem (Abiad-DB-De Bruyn-Koolen)

If a strictly Neumaier graph has five eigenvalues, then the number of closed walks of length 4 is at least $k\left(\frac{es-2e+1}{(s-1)^2}k^2+(s-e-1)\lambda\right)$.

Small eigenvalues

Remark

If k, θ_1 and θ_2 are the eigenvalues of a strongly regular graph, then $k > \theta_1 > 0 > \theta_2$.

Theorem (Seidel)

All strongly regular graphs with $\theta_2 \geq -2$ are classified.

Small eigenvalues

Remark

If k, θ_1 and θ_2 are the eigenvalues of a strongly regular graph, then $k > \theta_1 > 0 > \theta_2$.

Theorem (Seidel)

All strongly regular graphs with $\theta_2 \geq -2$ are classified.

Theorem (Doob-Cvetković, Abiad-De Bruyn-D'haeseleer-Koolen)

A strictly Neumaier graph has smallest eigenvalue smaller than -2.

23 The smallest eigenvalue and e

Theorem (Abiad-DB-De Bruyn-Koolen)

A strictly Neumaier graph with smallest eigenvalue $-m \leq -2$ has $e \leq m^5 - 5m^4 + 9m^3 - 5m^2 + 2m + 1.047$.

Open questions

24 General questions

Problem

Are there strictly Neumaier graphs with e not a power of 2?

Problem

Can we find Neumaier graphs with large e and large smallest eigenvalue (close to our bound)? Can the bound be improved?

Problem

Are there strictly Neumaier graphs with five eigenvalues?

Problem

Can we characterise strictly Neumaier graphs where the number of common neighbours of two non-adjacent vertices can only be one of two given values?

Questions about our new construction

Problem

Given an odd number q what do we know about $|S_{pq}(a) \cap (S_{pq}(a) + 1)| \pmod{q}$? Is there an infinite number of p and a for which it is -2?

Problem

Can we generalise this construction to other rings?

Q&A

Thank you for your attention

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Questions?