Constructions of new matroids and designs over $\mathbb{F}_q$

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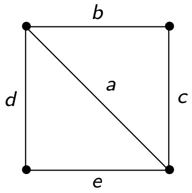
Matroid: a pair (E, r) with

- E finite set;
- $r : 2^E \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \in E$:
  - (r1) $0 \leq r(A) \leq |A|$
  - (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  - (r3) $r(A \cup B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Example

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{pmatrix}$$

Example



But: most matroids don't come from a matrix or graph.

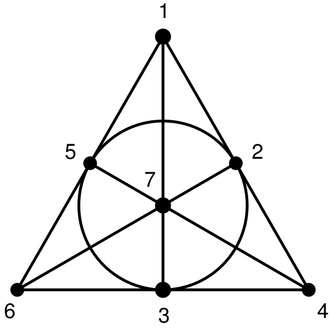
Independent set: subset with rank equal to cardinality

Circuit: minimal dependent subset

Flat: subset such that adding an element increases the rank

Matroids are completely determined by their independent sets, circuits, and flats.

Example (Fano plane)



$E = \{1, 2, 3, 4, 5, 6, 7\}$

Independent sets:
$\emptyset$, points, pairs of points,
3 points not on a line

Flats:
$\emptyset$, points, lines, E

Circuits:
lines, 4 points with no 3 on a line

A perfect matroid design (PMD) is a matroid where all flats of the same rank have the same size.

Example

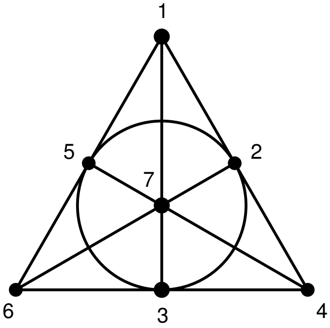
- Finite projective space and its truncations
- Finite affine space and its truncations
- Steiner systems
- Triffids, coming from finite commutative Moufang loops

Theorem (Murty, Young, Edmonds, 1970)

Given a PMD, we can maken the following designs:

- *All flats of cardinality j form the set of blocks of a design.*
- *All independent sets of cardinality j form the set of blocks of a design.*
- *All circuits of cardinality j form the set of blocks of a design.*

Example (Fano plane)



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Women in Numbers Europe, August 2019

q-Analogues

lattice	Boolean	subspace lattice of $\mathbb{F}_q^n$
atom	element	1-dim subspace
height	size	dimension
# atoms	n	$\frac{q^n-1}{q-1}$
meet $\wedge$	intersection	intersection
join $\vee$	union	sum

From q-analogue to 'normal': let $q \to 1$.

q-Matroid: a pair (E, r) with

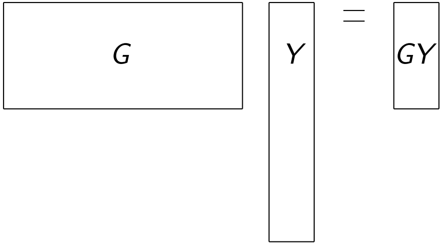
- E finite dimensional vector space;
- $r : \{\text{subspaces of } E\} \to \mathbb{N}_0$ a function, the *rank function*, with for all $A, B \subseteq E$:
  - (r1) $0 \leq r(A) \leq \dim A$
  - (r2) If $A \subseteq B$ then $r(A) \leq r(B)$.
  - (r3) $r(A + B) + r(A \cap B) \leq r(A) + r(B)$ (semimodular)

Theorem (J. & Pellikaan, 2018)

Every $\mathbb{F}_{q^m}$-linear rank metric code gives a q-matroid.

Proof.

Let $E = \mathbb{F}_q^n$ and G be a generator matrix of the code.
Let $A \subseteq E$ and Y a matrix whose columns span A.



Then $r(A) = \text{rk}(GY)$ satisfies the axioms (r1),(r2),(r3). □

A *q*-PMD is a *q*-matroid where all flats of the same rank have the same dimension.

Example

All *q*-Steiner systems are *q*-PMD's, where the blocks of the *q*-Steiner system are the maximal proper flats of the *q*-PMD.

Theorem (Byrne, Ceria, Ionica, J., Saçıkara, 2020)

Given a q-Steiner system and viewing it as a q-PMD, we can maken the following subspace designs:

- *All flats of dimension j form the set of blocks of a design.*
- *All independent spaces of dimension j form the set of blocks of a design.*
- *All circuits of dimension j form the set of blocks of a design.*

Corollary

There exists a $2\text{-}(13, 4, 5115; 2)$ *design.*

Theorem (Byrne, Ceria, Ionica, J., Saçıkara, 2020)
The subspace designs obtained from a q-Steiner system in the previous theorem have the same automorphism group as the q-Steiner system.

What we need: more q-Steiner systems!

Thank you for your attention!