Cameron-Liebler line classes in AG$(3, q)$

Joint work with J. D'haeseleer, L. Storme and A. Švob

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CDC Rijeka, July 14, 2021

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Introduction: Historical Context

Definition

A *Cameron-Liebler line class* $\mathcal{L}$ is a set of lines in $\mathrm{PG}(3, q)$, such that for every line spread $\mathcal{S}$ it holds that $|\mathcal{L} \cap \mathcal{S}| = x$. Here x denotes an integer which is called the parameter of $\mathcal{L}$.

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Other definitions:

- ... is a Boolean degree 1 function.
- ... for every line ℓ there are $q^2(x - \chi_{\mathcal{L}}(\ell))$ lines of $\mathcal{L}$ skew to ℓ.
- ... for every plane π and point $p \in \pi$, it holds that $|star(p) \cap \mathcal{L}| + |line(\pi) \cap \mathcal{L}| = x + (q+1)|pencil(p, \pi) \cap \mathcal{L}|$.

Introduction: Historical Context

Some basic observations in PG$(3, q)$

For a Cameron-Liebler line classes $\mathcal{L}$ and $\mathcal{L}'$ of parameters x and x' it holds that...

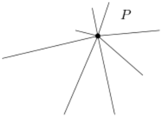
- $0 \leq x \leq q^2 + 1$
- $|\mathcal{L}| = x(q^2 + q + 1)$
- The complement of $\mathcal{L}$ is a Cameron-Liebler line class of parameter $q^2 + 1 - x$
- If $\mathcal{L} \cap \mathcal{L}' = \emptyset$, then $\mathcal{L} \cup \mathcal{L}'$ is a Cameron-Liebler line class of parameter $x + x'$.

Introduction: Historical Context

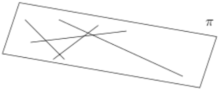
Example (of parameter $x = 0$)

the empty set.

Example (of parameter $x = 1$)



- All the lines through a fixed point p.
- All lines inside a plane.



Example (of parameter $x = 2$)

The set of lines inside a plane π union the lines through a point $p \notin \pi$.

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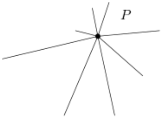

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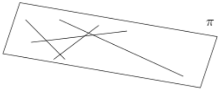

Example (of parameter $x = 2$)

The set of lines inside a plane π union the lines through a point $p \notin \pi$.

Theorem (Cameron and Liebler)

These examples and complements are the only examples for $x \in \{0, 1, 2, q^2 - 1, q^2, q^2 + 1\}$.

Introduction: Main research goals

1. Finding non-existence results for other parameters.
   Suppose that $\mathcal{L}$ is a Cameron-Liebler line class in $\mathrm{PG}(3, q)$ of parameter x.
   - (Govaerts and Storme) Then $x \leq 2$ or $x > q$.
   - (Metsch) Also $x > q\sqrt[3]{\frac{q}{2}} - \frac{2}{3}q$.
   - (Gavrilyuk and Metsch)Then for n equal to the number of lines of $\mathcal{L}$ in a plane or through a point, it holds that $x(x-1) + 2n(n-x) \equiv 0 \mod 2(q+1)$.
   - ...
2. Finding non-trivial examples.
   - (Bruen and Drudge) Cameron-Liebler line classes of parameter $x = \frac{q^2+1}{2}$ in $\mathrm{PG}(3, q)$, q odd.
   - (De Beule, Demeyer, Rodgers and Metsch) Cameron-Liebler line classes $\mathcal{L}$ of parameter $x = \frac{q^2-1}{2}$ in $\mathrm{PG}(3, q)$, $q \equiv 5$ or $9 \pmod{12}$.
   - ...

Cameron-Liebler Line Classes In AG$(3, q)$

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Definition

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Fact

There are more line spreads in AG$(3, q)$ then in PG$(3, q)$.

- (Type I) Every line spread of PG$(3, q)$ reduced to AG$(3, q)$.
- (Type II) All the lines through a point at infinity.
- ...

Cameron-Liebler Line Classes In AG$(3, q)$

Similar basic observations in AG$(3, q)$

For a Cameron-Liebler line classes $\mathcal{L}$ and $\mathcal{L}'$ of parameters x and x' it holds that...

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Example (of parameter $x = 1$)

All the lines through a fixed point p.

Example (of parameter $x = 2$)

???

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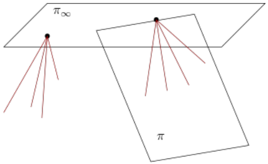

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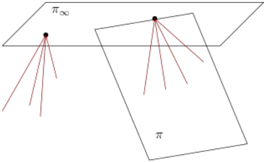

Example (of parameter $x = 2$)
???

> **Remark**
> There does not exist a Cameron-Liebler line class of parameter $x = 2$.

Cameron-Liebler Line Classes In AG$(3, q)$

Connection With PG$(3, q)$

Theorem

Every Cameron-Liebler line class in AG$(3, q)$ is a Cameron-Liebler line class in the projective closure PG$(3, q)$, and this line class has the same parameter x.

One can prove this by using the definitions and type I line spreads.

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Consequence

Every property of a Cameron-Liebler line class in PG$(3, q)$ can be translated to AG$(3, q)$.

Cameron-Liebler Line Classes In AG$(3, q)$

Modular Equality

Theorem (Gavrilyuk and Metsch)

Let $\mathcal{L}$ be a Cameron-Liebler line class of parameter x in PG$(3, q)$. Then for n equal to the number of lines of $\mathcal{L}$ in a plane or through a point, it holds that

$$x(x-1) + 2n(n-x) \equiv 0 \mod 2(q+1)$$

Cameron-Liebler Line Classes In AG$(3, q)$

Modular Equality

Corollary

Let $\mathcal{L}$ be a Cameron-Liebler line class of parameter x in AG$(3, q)$, then it holds that

$$x(x-1) \equiv 0 \mod 2(q+1).$$

Proof. Use the fact that $\mathcal{L}$ is a Cameron-Liebler line class in PG$(3, q)$ and the known modular equality to obtain that

$$x(x-1) + 2n(n-x) \equiv 0 \mod 2(q+1).$$

Here n is the number of lines of $\mathcal{L}$ through a certain point or in a certain plane. *In particular we may choose $n = 0$ or $n = x$.* □

Cameron-Liebler Line Classes In AG$(3, q)$

What if we choose another n?

Cameron-Liebler Line Classes In AG$(3, q)$

What if we choose another n?

Theorem (Choosing an affine plane)

For every (affine) plane π it holds that

$$|line(\pi) \cap \mathcal{L}| \equiv 0 \mod (q+1).$$

Proof. Use the fact that

$$|star(p) \cap \mathcal{L}| + |line(\pi) \cap \mathcal{L}| = x + (q+1)|pencil(p, \pi) \cap \mathcal{L}|,$$

with p at infinity! Hence $|star(p) \cap \mathcal{L}| = x$. □

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Conclusion.

$$x(x-1) + 2n(n-x) \equiv 0 \mod 2(q+1)$$

reduces to

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What if we choose another n?

Theorem (Choosing an affine point)

For every (affine) point p it holds that

$$|star(p) \cap \mathcal{L}| \equiv x \mod (q+1).$$

Proof. Use again the fact that

$$|star(p) \cap \mathcal{L}| + |line(\pi) \cap \mathcal{L}| = x + (q+1)|pencil(p, \pi) \cap \mathcal{L}|,$$

with π a plane. Hence $|line(\pi) \cap \mathcal{L}| = 0 \pmod{q+1}$. □

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with π a plane. Hence $|line(\pi) \cap \mathcal{L}| = 0 \pmod{q+1}$. □

Conclusion.

$$x(x-1) + 2n(n-x) \equiv 0 \mod 2(q+1)$$

reduces to

$$x(x-1) \equiv 0 \mod 2(q+1)$$

Modular Equality

Corollary

Let $\mathcal{L}$ be a Cameron-Liebler line class of parameter x in AG$(3, q)$, then it holds that

$$x(x-1) \equiv 0 \mod 2(q+1).$$

Question

Can we improve this equality by choosing another n?

Cameron-Liebler Line Classes In AG$(3, q)$

Modular Equality

Corollary

Let $\mathcal{L}$ be a Cameron-Liebler line class of parameter x in AG$(3, q)$, then it holds that

$$x(x-1) \equiv 0 \mod 2(q+1).$$

Question

Can we improve this equality by choosing another n?

No!

Cameron-Liebler Line Classes In AG$(3, q)$

Modular Equality

Consequence

- *There does not exist a Cameron-Liebler line class of parameter $x = 2$ in AG$(3, q)$.*

Consequence

- *If $2(q+1) = p_1^{h_1} \cdots p_s^{h_s}$, then the number of possible parameters of a Cameron-Liebler line class in AG$(3, q)$ is at most*

$$\begin{cases} 2^{s-1}q, & \text{if } q \text{ is even} \\ 2^{s-1}q - 2^{s-1} + 2, & \text{if } q \text{ is odd} \end{cases},$$

Proof. *Use the Chinese Remainder Theorem!* □

Cameron-Liebler Line Classes In AG$(3, q)$

Some Classification

Lemma

- *Classification of parameters in AG$(3,2)$:*
  $x \in \{0, 1, 3, 4\}$.
- *Classification of parameters in AG$(3,3)$:*
  $x \in \{0, 1, 8, 9\}$.
- *Classification of parameters in AG$(3,4)$:*
  $x \in \{0, 1, 15, 16\}$.
  Proof. $x \in \{5, 6, 10, 11\}$ *are ruled out by results of Govaerts and Penttila.* □

Cameron-Liebler Line Classes In AG$(3, q)$

Some Classification

Lemma

- *Possibilities of parameters in AG$(3,5)$:*
  $x \in \{0, 1, 12, 13, 24, 25\}$.

Proof. *Use the classification of Gavrilyuk and Metsch!*

$$x(x-1) \equiv 0 \mod 2(q+1).$$

□

Cameron-Liebler Line Classes In AG$(3, q)$

A Non-Trivial Example

Example (For $q \equiv 5$ or $9 \pmod{12}$.)

Consider the Cameron-Liebler line class in PG$(3, q)$ of parameter $x = \frac{q^2-1}{2}$ Given by found by De Beule, Demeyer, Rodgers and Metsch. Then this is **skew to the set of lines of a plane**, hence this line set is an example of a Cameron-Liebler line class in AG$(3, q)$.

A similar results was simultaneously found by Feng, Momihara and Xiang.

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Consequence

- *Classification of parameters in AG$(3, 5)$:* $x \in \{0, 1, 12, 13, 24, 25\}$.

Cameron-Liebler Line Classes In AG$(3, q)$

Classification of parameters and comparison

q	In PG$(3, q)$	In AG$(3, q)$
2	$x \in \{0, 1, 2, 3, 4, 5\}$	$x \in \{0, 1, 3, 4\}$
3	$x^{\ddagger} \in \{0, 1, 2, 5, 8, 9, 10\}$	$x \in \{0, 1, 8, 9\}$
4	$x^{\P} \in \{0, 1, 2, 7, 10, 15, 16, 17\}$	$x \in \{0, 1, 15, 16\}$
5	$x^{\S} \in \{0, 1, 2, 10, 12, 13, 14, 16, 24, 25, 26\}$	$x \in \{0, 1, 12, 13, 24, 25\}$

- ‡ : found by Drudge and Penttila
- ¶ : found by Gavrilyuk, Govaerts, Penttila and Mogilnyukh
- § : found by Gavrilyuk and Metsch

Thank you for your attention!

Are there any questions?

Cameron-Liebler Line Classes In AG$(3, q)$

Extra: A Second Connection With PG$(3, q)$

Theorem

Every Cameron-Liebler line class in PG$(3, q)$ that does not contain lines in a certain plane π, is a Cameron-Liebler line class of the same parameter in AG$(3, q)\cong$ PG$(3,q)/\pi$.

Proof. Is done by looking at the algebraic definition of Cameron-Liebler line classes in PG$(3, q)$. □

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