EIGENVALUES OF OPPOSITENESS GRAPHS AND ERDŐS-KO-RADO FOR FLAGS

joint work with Jan De Beule and Klaus Metsch

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Combinatorial Designs and Codes
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VRIJE
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Erdős-Ko-Rado problems

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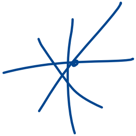
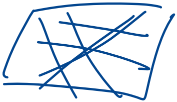
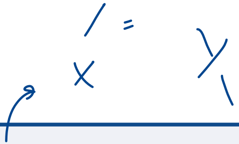
How large can a set of intersecting lines in $\mathrm{PG}(3, q)$ be?





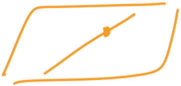
$q^2 + q + 1$

Erdős-Ko-Rado problems



How large can a set of intersecting lines in $\mathrm{PG}(3,q)$ be?

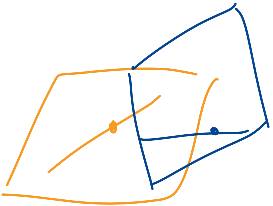
How large can a set of flags in $\mathrm{PG}(3,q)$ be?



Erdős-Ko-Rado problems

How large can a set of intersecting lines in $\mathrm{PG}(3, q)$ be?

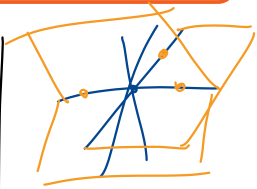
How large can a set of non-opposite flags in $\mathrm{PG}(3, q)$ be?



$P \notin \pi'$

$\ell \cap \ell' = \emptyset$

$\pi \not\ni P'$

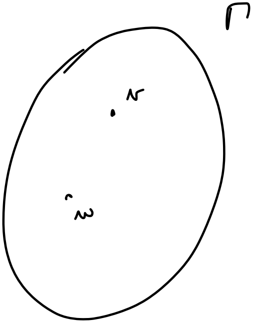


$(q^2+q+1)(q+1)^2$

The strategy

The strategy

1. Rephrase the problem



V(Γ) = lines

flags

v ~ w : non-intersection

oppositeness

→ look for cocliques

The strategy

1. Rephrase the problem

The strategy

1. Rephrase the problem

2. The ratio bound

Let Γ be a k-regular graph on n vertices whose adjacency matrix $A(\Gamma)$ has smallest eigenvalue λ. Then if C is a coclique we have

$$|C| \leq \frac{n}{1 - \frac{k}{\lambda}}.$$

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Let Γ be a k-regular graph on n vertices whose adjacency matrix $A(\Gamma)$ has smallest eigenvalue λ. Then if C is a coclique we have

$$|C| \leq \frac{n}{1 - \frac{k}{\lambda}}.$$

Moreover, if equality holds, then 1_C is contained in the sum of the eigenspaces corresponding to the eigenvalues k and λ.

The ratio bound

The ratio bound

$A(\Gamma)$ is contained in a symmetric association scheme.

↳ $J_q(4,2)$

- vector space of matrices over $\mathbb{C}$

- closed under $\cdot$

$\Longrightarrow$ commutative

The ratio bound

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$\Rightarrow$ coclique has size at most $\dfrac{\#\,\text{lines}}{1+q^2} = q^2 + q + 1$.

The ratio bound

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$A(\Gamma)$ in a non-commutative association scheme.

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$A(\Gamma)$ in a non-commutative association scheme.

$\to A(\Gamma)^2$

9.2.2. Theorem (Springer). *Let $\chi \in \mathrm{Irr}(\mathrm{KH})$. The element $\mathrm{T}_{w_0}^2$ is central in H and it acts on a simple module affording χ by the scalar*

$$z_\chi := \prod_{s \in S'} u_s^{f_s}, \quad \textit{where} \quad f_s := N_s\left(1 + \frac{\chi_1(s)}{\chi_1(1)}\right) \in \mathbb{Z}.$$

The ratio bound

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→ Geck - Pfeiffer

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$\Rightarrow$ coclique has size at most $\dfrac{\#\text{ flags}}{1+q^2} = (q^2+q+1)(q+1)^2$.

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Theorem (De Beule, M., Metsch)

We can compute the eigenvalues of opposition of

- *maximal flags in projective and polar spaces,*
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Corollary

We can derive EKR bounds for all these cases.

Galois geometries eSeminar
July 20, 16h CEST

Thank you for your attention!