Completely regular codes in Johnson and Grassmann graphs with small covering radii

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- Definitions and Introduction.
- Completely regular codes from affine SQS and Desarguesian 2-spreads.
- Binary linear programming for completely regular codes in $J_2(6,3)$ and the table of completely regular codes in $J_2(6,3)$ with covering radius 1.

The Johnson graph $J(n,k)$

The vertices are k-subsets of the set $\{1,\ldots,n\}$ and
the edges are subsets meeting in a $(k-1)$-subset.

The Grassmann graph $J_q(n,k)$

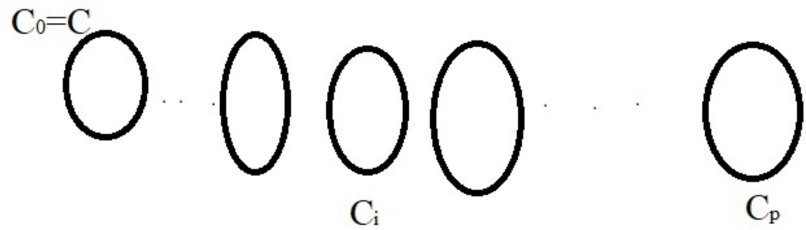
The vertices are k-subspaces of F_q^n and
the edges are pairs of subspaces meeting in a $(k-1)$-subspace.

Johnson graphs are a "part"of Grassmann series for $q \to 1$.
The Johnson graph $J(n,k)$ is denoted by $J_1(n,k)$.

For a code C in a graph Γ consider the distance partition with respect to C:

$$C_i = \{x \in V(\Gamma) : d(x, C) = i\},$$

$0 \leq i \leq \rho$, where $\rho = max_{x \in V(\Gamma)}(d(x, C))$ is the *covering radius* of C.



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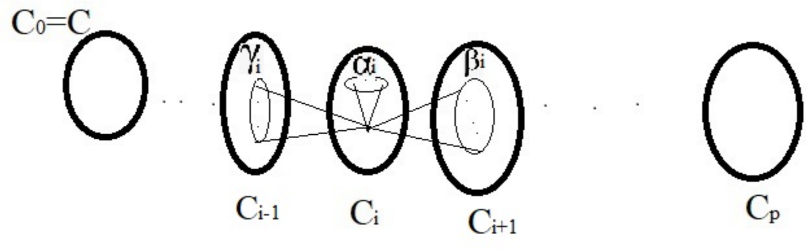
$$C_i = \{x \in V(\Gamma) : d(x, C) = i\},$$

$0 \leq i \leq \rho$.
A code C in Γ is called *completely regular* (shortly CRC) if $\exists$ $\alpha_0, \dots, \alpha_\rho$, $\beta_0, \dots, \beta_{\rho-1}$, $\gamma_1, \dots, \gamma_\rho$ such that for any $i \in \{0, \dots, \rho\}$ any vertex of C_i is adjacent to α_i, β_i and γ_i vertices of C_i, C_{i+1} and C_{i-1} respectively.



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$\alpha_0, \dots, \alpha_\rho$ can be found from β's and γ's and the valency of the graph.
The set $\{\beta_0, \dots, \beta_{\rho-1}; \gamma_1, \dots, \gamma_\rho\}$ is called the *intersection array* of the completely regular code C.

Designs and q-ary designs

A collection D of k-subspaces (k-subsets when $q = 1$) of F_q^n (of $\{1, \ldots, n\}$) is a $t-(n, k, \lambda)_q$-*design*, if any t-subspace of F_q^n (t-element subset of $\{1, \ldots, n\}$) is contained in exactly λ elements of D.

No repeated blocks, so designs are codes in $J_q(n, k)$, $q \geq 1$.

The *strength* of D is the maximum t such that D is a t-design.

For $q \geq 2$, a $1-(n, k, 1)_q$-design is called a k-*spread*.

Strength from intersection array

The *strength* of D is the maximum t such that D is a t-design.

The strength of CRCs in Johnson/Grassmann graphs could be calculated from their intersection array [Martin, 98].

Completely regular codes in Hamming graphs

CRCs in Hamming graphs include: perfect, some BCH, Preparata codes etc.

[Borges, Zinoviev, Rifa]: A survey on CRCs in Hamming and Johnson graphs.

Some CRCs in Grassmann and Johnson graphs

The CRCs of strength 0 and $\rho = 1$ in the Grassmann graphs $J_q(n, 2)$ coincide with Cameron-Liebler line classes in $PG(n-1, q)$.

- any $(k-1)-(n, k, \lambda)_q$-design is a CRC in $J_q(n, k)$, $q \geq 1$ [Delsarte] [Martin '98] [Avgustinovich, Potapov].

Theorem [Avgustinovich, M.]

Let D be a $(k-1)-(n, k, 1)$-design. Then

$$\{x : |x| = k+1, |\{y \in D : y \subset x\}| = 0\}$$

is completely regular in $J(n, k+1)$ with $\rho = 1$.

The theorem hold for the Grassmann graphs as well, but not many $(k-1)-(n, k, 1)_q$-design are known yet: 2-spreads and $2-(13, 3, 1)_2$-design [Braun, Etzion, Ostergard, Vardy, Wassermann].

A new series from affine SQS in Johnson graph

Previous results:

- Any Steiner quadruple system D of order n is a CRC in $J(n, 4)$ with $\rho = 1$ [Martin 98].

Theorem [Avgustinovich, M.]

If D is a Steiner quadruple system of order n, then $\{x : |x| = 5, |\{y \in D : y \subset x\}| = 0\}$ is a completely regular code in $J(n, 5)$ with $\rho = 1$.

We can go one step further in a special case:

Theorem

If D is the affine SQS (affine 2-subspaces of F_2^m) of order 2^m, then $\{x : |x| = 6, |\{y \in D : y \subset x\}| = 0\}$ is a completely regular code with $\rho = 2$ in $J(2^m, 6)$.

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A new series from Desarguesian spread in Grassmann graph

- Any 2-spread D is a CRC in $J_q(n, 2)$ with $\rho = 1$ [Avgustinovich, Potapov], [Martin 98].
- If D is 2-spread, then

$$\{x : dim(x) = 3, |\{y \in D : y < x\}| = 0\}$$

is a CRC in Grassmann graph $J_q(n, 3)$ with $\rho = 1$. This fact follows from [Avgustinovich, M.], see also [De Winter, Metsch].

Theorem

Let D be a Desarguesian 2-spread. Then

$$\{x : dim(x) = 4, |\{y \in D : y < x\}| = 0\}$$

is a completely regular code in $J_q(n, 4)$ with $\rho = 2$.

Proposition

A code C is completely regular with $\rho = 1$ and intersection array $\{\beta_0; \gamma_1\}$ in a m-regular graph with adjacency matrix A iff its characteristic vector χ_C fulfills

$$A\chi_C = (m - \beta_0)\chi_C + \gamma_1(\overline{\mathbf{1}} - \chi_C).$$

Here $\overline{\mathbf{1}}$ is the all-ones vector.

0-1 linear programming for completely regular codes:
[Bamberg] J. Bamberg, There is no Cameron–Liebler line class of $PG(3,4)$ with parameter 6,
http://symomega.wordpress.com/2012/04/01/there-is-no-cameron-liebler-line-class-of-pg34-with-parameter-6

Some CRCs (q-ary 1-designs) in $J_2(6,3)$ could be found using binary linear programming :

Theorem

There are completely regular codes in $J_2(6,3)$ with covering radius 1, intersection array $\{93-\gamma_1;\gamma_1\}$ for any $\gamma_1 \in \{12,15,18,24,27,30\}$. All codes are 1-designs.

The codes are invariant under the following mapping:

$$f : F_{2^6} \to F_{2^6}$$

$$f(b) = b \cdot a^{21},$$

where a is a primitive element of F_{2^6}.

Таблица: Completely regular codes in $J_2(6,3)$ with $\rho = 1$. The intersection array $\{\beta_0, \gamma_1\}$ of a CRC is obtained from its strength s and γ_1: $98 - \beta_0 - \gamma_1 = 2^{s+1}([{3-s-1 \atop 1}]_q)^2 - [{s+1 \atop 1}]_q$

Design strength	Integer conditions	Nonexis-, tence, γ_1	Existence, γ_1	Open cases, γ_1
0	$\gamma_1 \mod 7 = 0$	$21^{FI}, 28^{FI}$	$7^H, 14^{HP}$	
1	$\gamma_1 \mod 3 = 0$	$3^{M''}$	$9^M, 21^{M'}$, $3l, l = 4,5,6$ $8, 9, 10^A$	$3l$, $l \in \{2\} \cup \{10, \ldots, 15\}$
2	$\gamma_1 \mod 21 = 0$		$21^B, 42^B$	

^{H} subspaces belonging to a hyperplane;
^{HP} subspaces are in a hyperplane H or contain a vector v, $v \notin H$;
^{A} obtained using $0-1$ linear programming;
^{B} correspond to $2-(6,3,3)_2-$ and $2-(6,3,6)_2$-designs [Braun, Kerber, Laue];
^{M} totally isotropic subspaces of a symplectic polarity [De Winter, Metsch];
$^{M'}$ the code $\{y : y < F_2^6, dim(y) = 3, |\{x : x \in C, x < y\}| = 0\}$, where C is a 2-spread;
$^{M''}$ nonexistence follows from [Lemma 21, De Winter, Metsch].

^{FI} correspond to Cameron-Liebler sets; nonexistence was proven in [Filmus, Ihringer].

These results are available as preprint:
Completely regular codes in Johnson and Grassmann graphs with small covering radii, I. Yu. Mogilnykh, arxiv.org/abs/2012.06970

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Thank you for your attention