Quantum Solution to the Problem of 36 Officers of Euler

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[1] several pictures used here are taken directly from Grzegorz's former presentation

e-Print

https://arxiv.org/abs/2104.05122

Graeco-Latin squares (d = 3)

given $\{A, B, C\}$ and $\{\alpha, \beta, \gamma\}$

arrange them into Latin squares
(each row and column includes single symbol)

A	B	C
C	A	B
B	C	A

$\cup$

α	β	γ
β	γ	α
γ	α	β

$=$

A, α	B, β	C, γ
C, β	A, γ	B, α
B, γ	C, α	A, β

mix them $\rightarrow$ the last square contains distinct pairs (Latin, Greek)

Graeco-Latin squares

ranks and suits

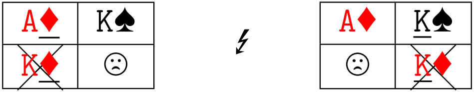
$$\{A, B, C\} \to \{\mathtt{A}, \mathtt{K}, \mathtt{Q}\} \text{ and } \{\alpha, \beta, \gamma\} \to \{\diamondsuit, \spadesuit, \clubsuit\}$$

$\ldots =$

A♦	K♠	Q♣
Q♠	A♣	K♦
K♣	Q♦	A♠

Graeco-Latin squares • Existence Problem

take { A♠, A♦, K♠, K♦ }



Graeco-Latin squares
a. k. a.
orthogonal Latin squares (OLS) exist [2] [3] [4] $\forall d \in \mathbb{N} \setminus \{2, 6\}$

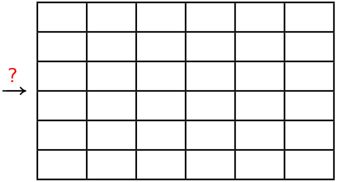
[2] G. Tarry; Compte Rendu... **1**: 122–123 and **2**: 170–203 (1900)
[3] R.C. Bose, S.S. Shrikhande, E.T. Parker; Can. J. Math. **12**: 189–203 (1960)
[4] C.J. Colbourn, J.H. Dinitz; J. Stat. Planning Inference **95**: 9 (2001)

Euler's Problem (d = 6)

"[...] The question revolves around arranging 36 officers
to be drawn from 6 different regiments so that they are ranged in a square
so that in each line (both horizontal and vertical)
there are 6 officers of different ranks and different regiments."

– Leonhard Euler[5]

A♠	K♠	Q♠	J♠	10♠	9♠
A♣	K♣	Q♣	J♣	10♣	9♣
A♥	K♥	Q♥	J♥	10♥	9♥
A♦	K♦	Q♦	J♦	10♦	9♦
A✿	K✿	Q✿	J✿	10✿	9✿
A✶	K✶	Q✶	J✶	10✶	9✶



[5] L. Euler; Recherches sur une nouvelle espece de quarres magiques (1779)

Approximate Solution

almost[6] OLS(6)

A♠	K♣	Q♦	J♥	10✿	9✶
K♦	A♥	J✿	Q✶	9♠	10♣
Q♣	J♠	9♥	10♦	A✶	K✿
J✶	Q✿	10♠	9♣	K♥	A♦
10♥	9♦	K✶	A✿	J♣	Q♠
9✿	10✶	A♣	K♠	Q♦	J♥

[6] L. Clarisse, S. Ghosh, S. Severini, A. Sudbery; Phys. Rev. **A** 72, 012314 (2005)

Classical to Quantum Transition

C: probability vector → Q: normalized complex state $|\text{vector}\rangle \in \mathcal{H}^d$

quantum Latin square[7] of order d
$\equiv d \times d$ array of elements of the Hilbert space $\mathcal{H}^d$
such that every row and every column is an **orthonormal** basis

example:

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\frac{\|1\rangle-\|2\rangle}{\sqrt{2}}$	$\frac{i\|0\rangle+2\|3\rangle}{\sqrt{5}}$	$\frac{2\|0\rangle+i\|3\rangle}{\sqrt{5}}$	$\frac{\|1\rangle+\|2\rangle}{\sqrt{2}}$
$\frac{\|1\rangle+\|2\rangle}{\sqrt{2}}$	$\frac{2\|0\rangle+i\|3\rangle}{\sqrt{5}}$	$\frac{i\|0\rangle+2\|3\rangle}{\sqrt{5}}$	$\frac{\|1\rangle-\|2\rangle}{\sqrt{2}}$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

ket $|k\rangle = \hat{e}_k = [0, \ldots, \overset{(k+1)}{1}, \ldots, 0]^{\mathrm{T}}$, e.g. $i|0\rangle + 2|3\rangle = [i, 0, 0, 2]^{\mathrm{T}} \in \mathbb{C}^4$

[7] B. Musto, J. Vicary; Quant. Inf. Comp. **16**: 1318–1332 (2016)

QOLS

Q_1 and Q_2 are **quantum orthogonal Latin squares** (QOLS) iff the pointwise inner product of any row from Q_1 with any row from Q_2 yielding a single 1 and with the rest being 0

example:

$\|0\rangle$	$\|1\rangle$	$\|2\rangle$	$\|3\rangle$
$\|1\rangle$	$\|0\rangle$	$\|3\rangle$	$\|2\rangle$
$\|2\rangle$	$\|3\rangle$	$\|0\rangle$	$\|1\rangle$
$\|3\rangle$	$\|2\rangle$	$\|1\rangle$	$\|0\rangle$

$\perp$

$\|0\rangle$	$\|2\rangle$	$\|3\rangle$	$\|1\rangle$
$\|1\rangle$	$\|3\rangle$	$\|2\rangle$	$\|0\rangle$
$\|2\rangle$	$\|0\rangle$	$\|1\rangle$	$\|3\rangle$
$\|3\rangle$	$\|1\rangle$	$\|0\rangle$	$\|2\rangle$

see: `http://qpl2016.cis.strath.ac.uk/pdfs/3Musto.pdf` /retrieved 2021-07-09/

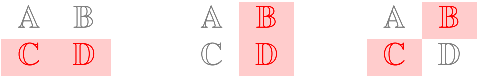
* * * * * * * *

question: are there QOLS(6)?

if so, there exists a solution to the quantum version of the Euler's problem

Quantum Entanglement in QIT

- **entanglement** = strong correlation between physical systems
- $\mathcal{H}_A \otimes \mathcal{H}_B = \mathcal{H}_{AB} = \{a \otimes b\}_{\text{sep.}} \cup \left\{ \sum_{jk} a_j \otimes b_k \right\}$
- measures of entanglement... → **maximally entangled states**
- four-partite physical system $\mathbb{S} = \mathbb{A}_6 \otimes \mathbb{B}_6 \otimes \mathbb{C}_6 \otimes \mathbb{D}_6$ (local dimension = 6)
- three possible (even) bi-partitions of four-partite system



(this originates from deeper physical consideration)

- **Absolutely Maximally Entangled** states
  ≡ maximally entangled w.r.t. every possible even bi-partition
- AME(4, 6) - four subsystems with six degrees of freedom each

(four particles, each of 6 possible energy levels)

AME(4,6) • Unitary Representation

evolution of (isolated) quantum system is governed by **unitary operators**

- 4−partite system $\Longleftrightarrow$ tensor structure of the form $\mathbb{U}(6) \otimes \mathbb{U}(6)$
- **partial transpose** (transpose within each block); $U^{\Gamma}_{abcd} = U_{adcb}$

$\mathbb{U}(36) \ni U =$

B_{11}	B_{12}	B_{13}	B_{14}	B_{15}	B_{16}
B_{21}	B_{22}	B_{23}	B_{24}	B_{25}	B_{26}
$\vdots$					
B_{61}	B_{62}	B_{63}	B_{64}	B_{65}	B_{66}

$\xrightarrow{\Gamma}$

B^{T}_{11}	B^{T}_{12}	B^{T}_{13}	B^{T}_{14}	B^{T}_{15}	B^{T}_{16}
B^{T}_{21}	B^{T}_{22}	B^{T}_{23}	B^{T}_{24}	B^{T}_{25}	B^{T}_{26}
$\vdots$					
B^{T}_{61}	B^{T}_{62}	B^{T}_{63}	B^{T}_{64}	B^{T}_{65}	B^{T}_{66}

- **reshuffling** (exchange rows and blocks); $U^{R}_{abcd} = U_{acbd}$

$U =$

$u_{11} \cdots u_{16}$ $\vdots \quad \vdots$ $u_{61} \cdots u_{66}$	next block					
$\vdots$						$\vdots$
						last block

$\xrightarrow{R}$

$u_{11} \cdots u_{16}$ next unfolded_ $\vdots$	· _	· _	· _	· _	$u_{61} \cdots u_{66}$ _block
$\vdots$					$\vdots$
last unfolded_	_	_	_	_	_block

AME(4,6) • Semi-Analytical Algorithm

- equivalent problem

$\mathbb{A}\ \mathbb{B}\ /\ \mathbb{C}\ \mathbb{D} \leftrightarrow U$ $\qquad$ $\mathbb{A}\ \mathbb{B}\ /\ \mathbb{C}\ \mathbb{D} \leftrightarrow U^{\Gamma}$ $\qquad$ $\mathbb{A}\ \mathbb{B}\ /\ \mathbb{C}\ \mathbb{D} \leftrightarrow U^{\mathrm{R}}$

- **GOAL: find a matrix $\mathcal{U}$ such that**
  $\mathcal{U} \in \mathbb{U}(36)$, $\mathcal{U}^{\Gamma} \in \mathbb{U}(36)$ **and** $\mathcal{U}^{\mathrm{R}} \in \mathbb{U}(36)$ (they are called 2−unitary)
- numerics: non-linear contractive map $\mathcal{M}_{\Gamma\mathrm{R}} : \mathbb{C}^{36\times 36} \to \mathbb{C}^{36\times 36}$
  U_0 = appropriate seed (starting point)
  iterative procedure:
  $$U_0 \overset{\Gamma}{\to} U_1 \overset{\mathrm{R}}{\to} U_2 \overset{\text{Polar Decomp.}}{\to} U_3 \overset{\Gamma}{\to} U_4 \overset{\mathrm{R}}{\to} U_5 \overset{\mathrm{PD}}{\to} U_6 \to \cdots \to U_\infty$$
- **for some seeds, map $\mathcal{M}_{\Gamma\mathrm{R}}$ converges to a 2−unitary matrix $\mathcal{U}$**
- local unitary rotations preserve 2−unitarity
  $(U_A \otimes U_B)\mathcal{U}_{(\mathtt{ugly})}(U_C \otimes U_D) = \mathcal{U}_{(\mathtt{nice})}$ for $U_X \in \mathbb{U}(6)$

Absolutely Maximally Entangled State of Four Quhexes

|A♣⟩ |K♠⟩	|K♥⟩ |A♦⟩	|9✿⟩ |10✷⟩ |10✿⟩ |9✷⟩	|10♣⟩ |9♠⟩ |10♠⟩ |9♣⟩	|J♥⟩ |Q♦⟩ |Q♥⟩ |J♦⟩	|Q✿⟩ |J✷⟩ |Q✷⟩ |J✿⟩
|9♠⟩ |10♣⟩	|10♦⟩ |9♥⟩	|J✿⟩ |Q✷⟩ |Q✿⟩ |J✷⟩	|Q♠⟩ |J♣⟩	|K♦⟩ |A♥⟩ |A♦⟩ |K♥⟩	|A✿⟩ |K✷⟩ |A✷⟩ |K✿⟩
|J✷⟩ |Q✿⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|K✿⟩ |A✷⟩ |A✿⟩ |K✷⟩	|9♣⟩ |10♠⟩	|10♥⟩ |9♦⟩ |10♦⟩ |9♥⟩
|A✷⟩ |K✿⟩ |A✿⟩ |K✷⟩	|K♣⟩ |A♠⟩ |A♣⟩ |K♠⟩	|9♥⟩ |10♦⟩ |10♥⟩ |9♦⟩	|10✿⟩ |9✷⟩ |10✷⟩ |9✿⟩	|J♠⟩ |Q♣⟩ |Q♠⟩ |J♣⟩	|Q♥⟩ |J♦⟩ |Q♦⟩ |J♥⟩
|9♦⟩ |10♥⟩	|10✷⟩ |9✿⟩	|J♣⟩ |Q♠⟩ |Q♣⟩ |J♠⟩	|Q♦⟩ |J♥⟩	|K✷⟩ |A✿⟩ |A✷⟩ |K✿⟩	|A♠⟩ |K♣⟩
|J♦⟩ |Q♥⟩	|Q✷⟩ |J✿⟩	|K♠⟩ |A♣⟩ |A♠⟩ |K♣⟩	|A♦⟩ |K♥⟩ |A♥⟩ |K♦⟩	|9✷⟩ |10✿⟩	|10♠⟩ |9♣⟩ |10♣⟩ |9♠⟩

- if every cell had only the largest |ket⟩, the solution would be classical ϟϟ
- officers come in pairs or double-pairs → entanglement!

AME(4,6) • Reconstruction Unitary U(36)

|A♣⟩ |K♠⟩	|K♥⟩ |A♦⟩	|9✿⟩ |10✷⟩ |10✿⟩ |9✷⟩	|10♣⟩ |9♠⟩ |10♠⟩ |9♣⟩	|J♥⟩ |Q♦⟩ |Q♥⟩ |J♦⟩	|Q✿⟩ |J✷⟩ |Q✷⟩ |J✿⟩
|9♠⟩ |10♣⟩	|10♦⟩ |9♥⟩	|J✿⟩ |Q✷⟩ |Q✿⟩ |J✷⟩	|Q♠⟩ |J♣⟩	|K♦⟩ |A♥⟩ |A♦⟩ |K♥⟩	|A✿⟩ |K✷⟩ |A✷⟩ |K✿⟩
|J✷⟩ |Q✿⟩	|Q♣⟩ |J♠⟩	|A♥⟩ |K♦⟩	|K✿⟩ |A✷⟩ |A✿⟩ |K✷⟩	|9♣⟩ |10♠⟩	|10♥⟩ |9♦⟩ |10♦⟩ |9♥⟩
|A✷⟩ |K✿⟩ |A✿⟩ |K✷⟩	|K♣⟩ |A♠⟩ |A♣⟩ |K♠⟩	|9♥⟩ |10♦⟩ |10♥⟩ |9♦⟩	|10✿⟩ |9✷⟩ |10✷⟩ |9✿⟩	|J♠⟩ |Q♣⟩ |Q♠⟩ |J♣⟩	|Q♥⟩ |J♦⟩ |Q♦⟩ |J♥⟩
|9♦⟩ |10♥⟩	|10✷⟩ |9✿⟩	|J♣⟩ |Q♠⟩ |Q♣⟩ |J♠⟩	|Q♦⟩ |J♥⟩	|K✷⟩ |A✿⟩ |A✷⟩ |K✿⟩	|A♠⟩ |K♣⟩
|J♦⟩ |Q♥⟩	|Q✷⟩ |J✿⟩	|K♠⟩ |A♣⟩ |A♠⟩ |K♣⟩	|A♦⟩ |K♥⟩ |A♥⟩ |K♦⟩	|9✷⟩ |10✿⟩	|10♠⟩ |9♣⟩ |10♣⟩ |9♠⟩

- for example (using ranks-suits or multi-indices)
  $\text{row}_{(1-1)\times 6+2} = |\psi_{12}\rangle = c\omega^{17}|\text{A}\diamondsuit\rangle + c\omega^{19}|\text{K}\heartsuit\rangle = c\omega^{17}|1\rangle\otimes|4\rangle + c\omega^{19}|2\rangle\otimes|3\rangle$

AME(4,6) as 2-Unitary Matrix

(1,1)	(2,2)	(1,2)	(2,1)	
$a\,\omega^{10}$	a	$b\,\omega^{15}$	$b\,\omega^{5}$	(6,3)
		c	c	(1,1)
c	c			(5,6)
$b\,\omega^{10}$	b	$a\,\omega^{5}$	$a\,\omega^{15}$	(4,2)

(1,3)	(2,4)	(1,4)	(2,3)	
$a\,\omega^{2}$	$a\,\omega^{14}$	$b\,\omega$	$b\,\omega^{5}$	(2,5)
		$c\,\omega^{5}$	$c\,\omega^{19}$	(3,3)
$c\,\omega^{17}$	$c\,\omega^{19}$			(1,2)
$b\,\omega^{14}$	$b\,\omega^{6}$	$a\,\omega^{3}$	$a\,\omega^{7}$	(6,4)

(1,5)	(2,6)	(1,6)	(2,5)	
$a\,\omega$	$a\,\omega^{19}$	$b\,\omega^{14}$	$b\,\omega^{16}$	(4,1)
$a\,\omega$	$a\,\omega^{3}$	$b\,\omega^{10}$	$b\,\omega^{4}$	(3,4)
$b\,\omega^{4}$	$b\,\omega^{18}$	$a\,\omega^{3}$	$a\,\omega^{9}$	(2,6)
$b\,\omega^{2}$	$b\,\omega^{8}$	$a\,\omega^{5}$	$a\,\omega^{15}$	(5,5)

(3,1)	(4,2)	(3,2)	(4,1)	
$a\,\omega^{4}$	$a\,\omega^{10}$	$b\,\omega^{17}$	$b\,\omega^{7}$	(4,5)
		$c\,\omega^{2}$	$c\,\omega^{2}$	(3,2)
$c\,\omega^{10}$	$c\,\omega^{6}$			(2,4)
$b\,\omega^{7}$	$b\,\omega^{13}$	$a\,\omega^{10}$	a	(5,3)

(3,3)	(4,4)	(3,4)	(4,3)	
a	a	$b\,\omega^{15}$	$b\,\omega^{15}$	(4,6)
		c	$c\,\omega^{10}$	(6,1)
c	$c\,\omega^{10}$			(5,4)
b	b	$a\,\omega^{5}$	$a\,\omega^{5}$	(1,5)

(3,5)	(4,6)	(3,6)	(4,5)	
$a\,\omega^{2}$	a	$b\,\omega^{19}$	$b\,\omega^{13}$	(2,3)
		$c\,\omega^{16}$	c	(6,2)
$c\,\omega^{8}$	$c\,\omega^{16}$			(3,1)
$b\,\omega^{14}$	$b\,\omega^{12}$	$a\,\omega$	$a\,\omega^{15}$	(1,6)

(5,1)	(6,2)	(5,2)	(6,1)	
$a\,\omega^{3}$	$a\,\omega^{7}$	b	b	(1,4)
		c	$c\,\omega^{10}$	(2,1)
$c\,\omega^{13}$	$c\,\omega^{7}$			(3,5)
$b\,\omega^{9}$	$b\,\omega^{13}$	$a\omega^{16}$	$a\omega^{16}$	(6,6)

(5,3)	(6,4)	(5,4)	(6,3)	
$a\,\omega^{12}$	$a\,\omega^{14}$	$b\,\omega^{15}$	$b\,\omega$	(3,6)
		$c\,\omega^{14}$	$c\,\omega^{10}$	(5,1)
$c\,\omega^{7}$	$c\,\omega^{19}$			(2,2)
$b\,\omega^{14}$	$b\,\omega^{16}$	$a\omega^{7}$	$a\omega^{13}$	(4,3)

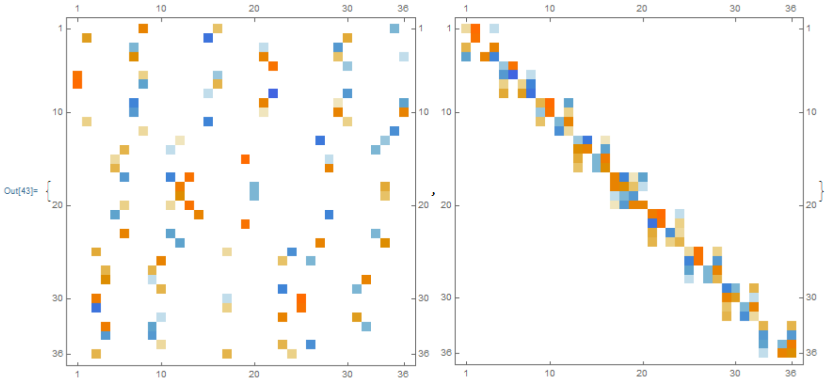
(5,5)	(6,6)	(5,6)	(6,5)	
$a\,\omega^{18}$	$a\,\omega^{18}$	$b\,\omega^{3}$	$b\,\omega^{3}$	(1,3)
		c	$c\,\omega^{10}$	(5,2)
$c\,\omega$	$c\,\omega^{11}$			(6,5)
$b\,\omega^{10}$	$b\,\omega^{10}$	$a\omega^{5}$	$a\omega^{5}$	(4,4)

$$\omega = \exp(i\pi/10), \qquad a = \frac{1}{\sqrt{5+\sqrt{5}}}, \qquad b = \sqrt{\frac{5+\sqrt{5}}{20}}, \qquad c = \frac{1}{\sqrt{2}} \qquad \Longrightarrow \qquad b/a = \varphi$$

AME(4,6) as 2-Unitary Matrix

- https://chaos.if.uj.edu.pl/~karol/Maestro7/data2.html

```
Y = PR.AME46.PC;
{AME46 // MatrixPlot, Y // MatrixPlot}
(*
LEFT: AME(4,6) as Unitary Matrix,
RIGHT: after permuting it reveals a block structure *)
```



Problems Solved

1. (unexpected) existence of **AME(4, 6) state**
2. $\Longrightarrow$ 2−**unitary** matrix of size 36
3. $\Longrightarrow$ two **QOLS** of size 6
4. $\Longrightarrow$ **perfect** tensor[8] with 4 indices $_{\{a,b,c,d\}}$ running from 1 to 6
5. $\Longrightarrow$ new classes of **quantum error correction codes** $((4, 1, 3))_6$ and shortened variant $((3, 6, 2))_6$
6. $\Longrightarrow$ ...

[8] Tensor T_{abcd} is **perfect** if any of its flattening into a matrix of order (here) 6^2 is unitary.

Problems to Solve

1. understanding the algorithm
2. other AME states
   `http://www.tp.nt.uni-siegen.de/+fhuber/ame.html`[9] /retrieved 2021-07-09/
3. equivalence relation → families of AME$(4,6)$?
4. real (orthogonal) case
5. will it help to solve the 6-dimensional MUB Problem?
   (Mutually Unbiased Bases in $\mathbb{C}^6$)

[9] F. Huber, N. Wyderka; Table of AME states

☺

Thank You!