Generalizations of Heffter arrays and biembedding (multi)graphs on surfaces

Simone Costa
University of Brescia

Joint work with F. Morini, A. Pasotti and M.A. Pellegrini

Combinatorial Designs and Codes
11 July - 16 July 2021

The results here presented are contained in the following papers:

S. Costa, F. Morini, A. Pasotti and M.A. Pellegrini, *A generalization of Heffter arrays*, J. Combin. Des. **28** (2020), 171–206.

S. Costa, A. Pasotti and M. A. Pellegrini, *Relative Heffter arrays and biembeddings*, Ars Math. Contemp. **18** (2020) 241–271.

S. Costa and A. Pasotti, *On λ-fold relative Heffter arrays and biembedding multigraphs on surfaces*, European J. Combin. **97** (2021) 103370.

Heffter Arrays

Definition (Archdeacon 2015)

Let $v = 2nk + 1$ be a positive integer. A square $\mathrm{H}(n;k)$ Heffter array over $\mathbb{Z}_v$ is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

a) *each row and each column contains k filled cells;*
b) *for every $x \in \mathbb{Z}_v \setminus \{0\}$, either x or $-x$ appears in the array;*
c) *the elements in every row and column sum to 0.*

Heffter Arrays

Definition (Archdeacon 2015)

Let $v = 2nk + 1$ be a positive integer. A square $\mathrm{H}(n; k)$ Heffter array over $\mathbb{Z}_v$ is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

a) *each row and each column contains k filled cells;*
b) *for every $x \in \mathbb{Z}_v \setminus \{0\}$, either x or $-x$ appears in the array;*
c) *the elements in every row and column sum to 0.*

Heffter arrays are combinatorial objects introduced by D.S. Archdeacon in 2015 in order to obtain:

- orthogonal k-cycle decompositions of complete graphs;
- biembeddings of complete graphs in orientable surfaces.

On the Existence Problem

- D.S. Archdeacon, T. Boothby and J.H. Dinitz, *Tight Heffter arrays exist for all possible values*, J. Combin. Des. **25** (2017), 5–35.
- D.S. Archdeacon, J.H. Dinitz, D.M. Donovan and E.S. Yazıcı, *Square integer Heffter arrays with empty cells*, Des. Codes Cryptogr. **77** (2015), 409–426.
- K. Burrage, N.J. Cavenagh, D. Donovan and E.S. Yazici, *Globally simple Heffter arrays $H(n;k)$ when $k \equiv 0, 3 \pmod 4$*, Discrete Math. **343** (2020), 111787.
- N.J. Cavenagh, J. Dinitz, D. Donovan and E.S. Yazici, *The existence of square non-integer Heffter arrays*, Ars Math. Contemp. **17** (2019), 369–395.
- J.H. Dinitz and I.M. Wanless, *The existence of square integer Heffter arrays*, Ars Math. Contemp. **13** (2017), 81–93.

On Biembeddings from Heffter

- D.S. Archdeacon, *Heffter arrays and biembedding graphs on surfaces*, Electron. J. Combin. **22** (2015) #P1.74.
- N.J. Cavenagh, D. Donovan and E.S. Yazici, *Biembeddings of cycle systems using integer Heffter arrays*, J. Combin. Des. **28** (2020), 900–922.
- S. Costa, F. Morini, A. Pasotti and M.A. Pellegrini, *Globally simple Heffter arrays and orthogonal cyclic cycle decompositions*, Austral. J. Combin. **72** (2018), 549–493.
- J.H. Dinitz and A.R.W. Mattern, *Biembedding Steiner triple systems and n-cycle systems on orientable surfaces*, Austral. J. Combin. **67** (2017), 327–344.

Relative Heffter Arrays

Definition (S.C., Morini, Pasotti, Pellegrini 2020)

Let $v = 2nk + t$ be a positive integer and let J be the subgroup of $\mathbb{Z}_v$ of order t. A square $\mathrm{H}_t(n; k)$ Heffter array over $\mathbb{Z}_v$ relative to J is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

- a) *each row and each column contains k filled cells;*
- b_1) *for every $x \in \mathbb{Z}_v \setminus J$, either x or $-x$ appears in the array;*
- c) *the elements in every row and column sum to 0.*

Relative Heffter Arrays

Definition (S.C., Morini, Pasotti, Pellegrini 2020)

Let $v = 2nk + t$ be a positive integer and let J be the subgroup of $\mathbb{Z}_v$ of order t. A square $\mathrm{H}_t(n; k)$ Heffter array over $\mathbb{Z}_v$ relative to J is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

a) *each row and each column contains k filled cells;*

b_1) *for every $x \in \mathbb{Z}_v \setminus J$, either x or $-x$ appears in the array;*

c) *the elements in every row and column sum to 0.*

- If $t = 1$, namely if J is the trivial subgroup of $\mathbb{Z}_{2nk+1}$, we find again the classical concept of Heffter array.

λ-fold Heffter Arrays

Definition (S.C., Pasotti 2021)

Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, and let J be the subgroup of $\mathbb{Z}_v$ of order t. A square ${}^{\lambda}\mathrm{H}_t(n;k)$ λ-fold Heffter array A over $\mathbb{Z}_v$ relative to J is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

a) *each row and each column contains k filled cells;*

b_2) *the multiset $\{\pm x \mid x \in A\}$ contains λ times each element of $\mathbb{Z}_v \setminus J$;*

c) *the elements in every row and column sum to 0.*

λ-fold Heffter Arrays

Definition (S.C., Pasotti 2021)

Let $v = \frac{2nk}{\lambda} + t$ be a positive integer, and let J be the subgroup of $\mathbb{Z}_v$ of order t. A square ${}^\lambda\mathrm{H}_t(n;k)$ λ-fold Heffter array A over $\mathbb{Z}_v$ relative to J is an $n \times n$ partially filled array with elements in $\mathbb{Z}_v$ such that:

a) *each row and each column contains k filled cells;*

b_2) *the multiset $\{\pm x \mid x \in A\}$ contains λ times each element of $\mathbb{Z}_v \setminus J$;*

c) *the elements in every row and column sum to 0.*

- If $\lambda = 1$, we find again the concept of relative Heffter array.

The following array A is a $\mathrm{H}_6(10;6)$ whose elements belong to $\mathbb{Z}_{126} \setminus \{0, \pm 21, \pm 42, 63\}$.

-1	5	2	-7	-9	10				
3	-4	-6	8	11	-12				
		-13	17	14	-19	25	-24		
		15	-16	-18	20	-23	22		
				-26	30	27	-32	-34	35
				28	-29	-31	33	36	-37
41	-45					-38	47	44	-49
-39	43					40	-46	-48	50
52	-57	-59	60					-51	55
-56	58	61	-62					53	-54

Let $\pi : \mathbb{Z}_{126} \to \mathbb{Z}_{63}$ be the natural projection. Then $\pi(A)$ is a ${}^2\mathrm{H}_3(10;6)$.

The following array A is a $\mathrm{H}_6(10;6)$ whose elements belong to $\mathbb{Z}_{126} \setminus \{0, \pm 21, \pm 42, 63\}$.

-1	5	2	-7	-9	10				
3	-4	-6	8	11	-12				
		-13	17	14	-19	25	-24		
		15	-16	-18	20	-23	22		
				-26	30	27	-32	-34	35
				28	-29	-31	33	36	-37
41	-45					-38	47	44	-49
-39	43					40	-46	-48	50
52	-57	-59	60					-51	55
-56	58	61	-62					53	-54

Let $\pi : \mathbb{Z}_{126} \to \mathbb{Z}_{63}$ be the natural projection. Then $\pi(A)$ is a ${}^2\mathrm{H}_3(10;6)$.

Remark (S.C., Pasotti 2021)

Given an ${}^{\alpha}\mathrm{H}_t(n;k)$ over $\mathbb{Z}_v$ and a divisor λ of t, we obtain an ${}^{\alpha\lambda}\mathrm{H}_{\frac{t}{\lambda}}(n;k)$ over $\mathbb{Z}_{v/\lambda}$ via projection map $\pi : \mathbb{Z}_v \to \mathbb{Z}_{v/\lambda}$.

Biembeddings

Definition

An embedding σ *of a (multi)graph* Γ *in a surface* Σ *is a continuous injective map between the topological representation of* Γ *and* Σ*.*

Definition

The connected components of $\Sigma \setminus \sigma(\Gamma)$ *are called faces of the embedding.*

Biembeddings

Definition

An embedding σ of a (multi)graph Γ in a surface Σ is a continuous injective map between the topological representation of Γ and Σ.

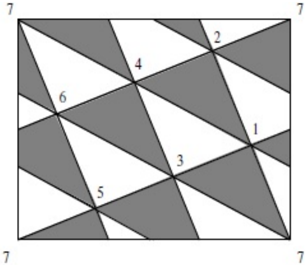
Definition

The connected components of $\Sigma \setminus \sigma(\Gamma)$ are called faces of the embedding.

Definition

A biembedding of a (multi)graph Γ in a surface Σ is a face 2-colorable embedding of Γ in Σ.

The following is a biembedding of K_7 in the torus.



Some Technical Requirements

Given a set $T \subseteq \mathbb{Z}_v$ and an ordering $\omega = (t_1, t_2, \ldots, t_k)$ of the elements in T, let $s_i = \sum_{j=1}^{i} t_j$.
We say that the ordering ω is simple (modulo v) if $s_i \neq s_h$ for $i \neq h$.

Definition

A ${}^{\lambda}\mathrm{H}_t(n; k)$ is said to be simple if its rows and columns admit a simple ordering (modulo $v = \frac{2nk}{\lambda} + t$).

Some Technical Requirements

Given a set $T \subseteq \mathbb{Z}_v$ and an ordering $\omega = (t_1, t_2, \ldots, t_k)$ of the elements in T, let $s_i = \sum_{j=1}^{i} t_j$.
We say that the ordering ω is simple (modulo v) if $s_i \neq s_h$ for $i \neq h$.

Definition

A $^{\lambda}\mathrm{H}_t(n;k)$ is said to be simple if its rows and columns admit a simple ordering (modulo $v = \frac{2nk}{\lambda} + t$).

Given a $^{\lambda}\mathrm{H}_t(n;k)$, we set

$$\omega_r = \omega_{R_1} \circ \ldots \circ \omega_{R_n};$$

$$\omega_c = \omega_{C_1} \circ \ldots \circ \omega_{C_m}.$$

Definition

Then ω_r and ω_c are said to be compatible if $\omega_c \circ \omega_r$ is a cycle of length nk.

Biembeddings of Complete Multipartite (Multi)Graphs

Theorem (S.C., Pasotti 2021; S.C., Pasotti, Pellegrini 2020)

Given a simple ${}^{\lambda}\mathrm{H}_t(n;k)$ *with respect to the compatible orderings* ω_r *and* ω_c. *Then there exists a biembedding of the graph* ${}^{\lambda}K_{(\frac{2nk}{\lambda t}+1)\times t}$ *in an orientable surface whose faces (boundary) are simple cycles of length* k.

Biembeddings of Complete Multipartite (Multi)Graphs

Theorem (S.C., Pasotti 2021; S.C., Pasotti, Pellegrini 2020)

Given a simple ${}^{\lambda}\mathrm{H}_t(n;k)$ *with respect to the compatible orderings* ω_r *and* ω_c*. Then there exists a biembedding of the graph* ${}^{\lambda}K_{(\frac{2nk}{\lambda t}+1)\times t}$ *in an orientable surface whose faces (boundary) are simple cycles of length* k*.*

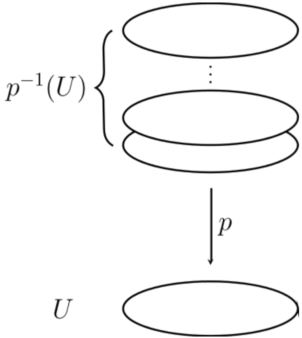
Theorem (S.C., Pasotti 2021; S.C., Pasotti, Pellegrini 2020)

There exists a biembedding of ${}^{\lambda}K_{(\frac{2nk}{\lambda t}+1)\times t}$ *in an orientable surface whose faces are simple cycles of length* k *in each of the following cases:*

λ	t	k	n
$\lambda = 1$	$t = k$	$k \in \{3,5,7,9\}$	$n \equiv 3 \pmod 4$
$\lambda = 2$	$t = 1$	$k = 3$	$n \equiv 1 \pmod 4$
$\lambda = 3$	$t = 1$	$k = 3$	$n \equiv 3 \pmod 4$
$\lambda = 2$	$t = 1$	$k = 5$	$n \equiv 3 \pmod 4$
$\lambda = 1$ *and* $\lambda \mid t$	$t \in \{n, 2n\}$	$k = 3$	$n \equiv 1 \pmod 2$

Covering Space

Let Σ' and Σ be topological spaces. We say that Σ' is a covering space for Σ if there exists a covering map $p : \Sigma' \to \Sigma$ such that:



A Topological Consideration

Let A be an ${}^{\alpha}\mathrm{H}_t(n;k)$ over $\mathbb{Z}_v$ (where $v = \frac{2nk}{\alpha} + t$) and let B be the ${}^{\alpha\lambda}\mathrm{H}_{\frac{t}{\lambda}}(n;k)$ obtained from A via projection map $\pi : \mathbb{Z}_v \to \mathbb{Z}_{v/\lambda}$.

A Topological Consideration

Let A be an ${}^{\alpha}\mathrm{H}_t(n;k)$ over $\mathbb{Z}_v$ (where $v=\frac{2nk}{\alpha}+t$) and let B be the ${}^{\alpha\lambda}\mathrm{H}_{\frac{t}{\lambda}}(n;k)$ obtained from A via projection map $\pi:\mathbb{Z}_v\to\mathbb{Z}_{v/\lambda}$.

Theorem (S.C., Pasotti 2021)

If B originates a biembedding $\sigma: {}^{\alpha\lambda}K_{(\frac{2nk}{\alpha t}+1)\times\frac{t}{\lambda}}\to\Sigma$, also A generates a biembedding $\sigma': {}^{\alpha}K_{(\frac{2nk}{\alpha t}+1)\times t}\to\Sigma'$ that is a covering space for Σ.

Denoted by p the covering map, the following diagram commutes:

$$\begin{tikzcd}
{}^{\alpha}K_{(\frac{2nk}{\alpha t}+1)\times t} \arrow[r, "\sigma'"] \arrow[d, "\pi"] & \Sigma' \arrow[d, "p"] \\
{}^{\alpha\lambda}K_{(\frac{2nk}{\alpha t}+1)\times\frac{t}{\lambda}} \arrow[r, "\sigma"] & \Sigma
\end{tikzcd}$$

Thanks for your attention