On the Hadamard maximal determinant problem

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Hadamard's determinant bound

Theorem (Hadamard, 1893)

Let M be an $n \times n$ matrix with complex entries of norm at most 1. Then

$$|\mathrm{det}(M)| \leq \sqrt{n}^n.$$

J. Hadamard, Résolution d'une question relative aux déterminants, Bull. Sci. Math., 17, 1893, 240–246.

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This bound will be saturated when

- Every entry is of norm 1, and
- The rows of M are pairwise orthogonal.

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The maximal determinant problem

Question

What is the maximal determinant of an $n \times n$ matrix with entries in $\{\pm 1\}$?

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Proposition

Suppose that H is a real matrix saturating the determinant bound. Then:

1. *All entries of H belong to $\{\pm 1\}$.*
2. *The rows and columns of H are orthogonal.*
3. *The order of H is 1, 2 or a multiple of 4.*

What about when $n \not\equiv 0 \mod 4$?

Existence of Hadamard matrices

- 2^t for $t \geq 0$. (Sylvester)
- $p^a + 1$ where p is prime and $p^a \equiv 3 \mod 4$. (Paley)
- $2(p^a + 1)$ where p is prime and $p^a \equiv 1 \mod 4$. (Paley)
- $p(p+2)+1$ where p and $p+2$ are twin primes. (Sprott, Stanton)
- $4p^{4t}$ where p is prime and $t \geq 1$. (Xia)
- $4t$ for all values of $t \leq 250$ except for $t \in \{167, 179, 223\}$. (Kharaghani, Tayfeh-Rezaie)
- $n = ab/2$ or $n = abcd/16$ where a, b, c, d are orders of Hadamard matrices. (Craigen, Seberry, Yamada, Zhang)

The Paley matrices

Let p be a prime number and χ be the quadratic character of $\mathbb{F}_p$. We define $\chi(0) = 0$. Then the *Paley core* matrix

$$Q = (\chi(x - y))_{0 \le x, y \le p-1}$$

has zeroes on the diagonal and off-diagonal entries in $\{\pm 1\}$. Further, Q is circulant and satisfies $QQ^\top = pI - J$.

R.E.A.C. Paley, On orthogonal matrices, J. Math. Phys., 12, 1933, 311–320.

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Proposition (Paley)

Suppose that $p \equiv 3 \mod 4$ is prime, and let j_p denote the column vector of length p of all ones. Then the matrix

$$M = \begin{bmatrix} Q + I & -j_p \\ j_p^\top & 1 \end{bmatrix}$$

is a (skew-symmetric) maximal determinant matrix of order $p + 1$.

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Look at the Gram matrix

Theorem (Ehlich, Wojtas)

Let G be an $n \times n$ symmetric positive definite matrix, with diagonal entries n and $|g_{i,j}| \geq b$ for all $i \neq j$. If $\det(G) = (n + (n-1)b)(n-b)^{n-1}$, then up to permutation and negation of rows and columns,

$$G = (n-b)I + bJ,$$

where J is the all-ones matrix.

H. Ehlich, Determinantenabschätzungen für binäre Matrizen, Math. Z., 83, 1964, 123–132.

M. Wojtas, On Hadamard's inequality for the determinants of order non-divisible by 4, Colloq. Math., 12, 1964, 73–83.

The Barba bound

Theorem (Barba)

Let M be a matrix of odd order with entries in $\{\pm 1\}$. Then $\det(M) \leq \sqrt{2n-1}(n-1)^{\frac{n-1}{2}}$.

G. Barba, Intorno al teorema di Hadamard sui determinanti a valore massimo, Giorn. Mat. Battaglini, III. Ser., 71, 1933, 70–86.

The Barba bound

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Theorem

Let M be an $n \times n$ matrix with entries in $\{\pm 1\}$. If $\det(M)$ *meets the Barba bound with equality then*:

- $2n-1$ *is a perfect square and* $n \equiv 1 \mod 4$.
- *Up to permutation and negation of rows and columns,* $MM^\top = (n-1)I + J$.

G. Barba, Intorno al teorema di Hadamard sui determinanti a valore massimo, Giorn. Mat. Battaglini, III. Ser., 71, 1933, 70–86.

Building blocks

Let M_p be the incidence matrix of the affine plane of order p.

Let $C = Q - I$, where Q is the *Paley core* of order p.

Let $M = M_p\left(I_{p+1} \otimes C\right)$ is a $p^2 \times (p^2 + p)$.

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M has entries in $\{\pm 1\}$ and satisfies

$$MM^\top = p^2 I_{p^2}.$$

It will be convenient to write M as a block matrix, which we denote $[M_0 \mid M_1]$ where M_0 is $p^2 \times p$.

When $n \equiv 1 \mod 4$

Theorem (Neubauer-Radcliffe)

Let W be the following matrix:

$$W = \begin{bmatrix} 1 & j_p & -j_{p^2} & j_p & j_{p^2} \\ j_p^\top & -J & -C \otimes j_p & J & (C+2I) \otimes j_p \\ j_{p^2}^\top & -j_p^\top \otimes C & -(C+I) \otimes C + I \otimes J & j_p^\top \otimes C & (C+I) \otimes C + I \otimes J \\ j_p^\top & J & -j_p \otimes C & J & j_p \otimes C \\ -j_{p^2}^\top & -M_0 & -M_1 & -M_0 & -M_1 \end{bmatrix}.$$

Then $WW^\top = (2p^2 + 2p)I + J$, and so W is a maximal determinant matrix.

M. G. Neubauer, A. J. Radcliffe, The maximum determinant of ± 1 matrices, Linear Algebra Appl., 257, 1997, 289–306.

When $n \equiv 3 \mod 4$

Theorem (Ehlich)

For $n \equiv 3 \mod 4$ and $n \geq 63$, an explicit upper bound on the maximal determinant of an $n \times n$ matrix M is

$$\det(MM^{\top}) \leq \frac{4 \cdot 11^6}{7^7} n(n-1)^6(n-3)^{n-7}.$$

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- The Barba bound is sharper for $n \leq 59$.
- No matrices known achieve this bound.
- Asymptotically optimal up to a constant factor.
- Lots of computational work done.

When $n \equiv 3 \mod 4$

Proposition (BEHÓC)

Suppose that R *and* S *are* $k \times k$ *matrices satisfying the identities*

$$RJ = JR = rJ, \quad SJ = JS = sJ, \quad RR^\top + SS^\top = (2k-2)I + 2J.$$

Let

$$M_1 = \begin{pmatrix} R & S & j_k^\top \\ S & -R & -j_k^\top \\ j_k & j_k & 1 \end{pmatrix}, \quad M_2 = \begin{pmatrix} R & S & j_k^\top \\ S^\top & -R^\top & -j_k^\top \\ j_k & j_k & 1 \end{pmatrix}.$$

Then

$$\det(M_i M_i^\top) = (4k^2r^2 - 16k^2r + 16k^2 - 16k + 8kr + 4)(2k-2)^{2k-2}$$

with the condition that $RS^\top = SR^\top$ *for* M_1 *and no additional condition for* M_2.

When $n \equiv 3 \mod 4$

Theorem

Let M be a matrix of order $n = 2k+1$ as in the previous Proposition, with $r^2 + s^2 = 4k - 2$. Then $\det(M)$ achieves a fraction at least $r^2/3n$ of the Ehlich bound.

- *A matrix exceeding* 0.34 *of the Ehlich bound exists of order $n = 4q^2 + 4q + 3$ for each prime power $q \geq 379$. A matrix exceeding $\frac{1}{3}$ of the bound exists for each $q \geq 47$.*
- *By a Theorem of Spence, a matrix exceeding* 0.48 *of the Ehlich bound exists of order $n = 2q^2 + 2q + 3$ for each $q \geq 233$. A matrix exceeding* 0.47 *of the bound exists for each $q \geq 43$.*

E. Spence, Skew-Hadamard matrices of the Goethals-Seidel type. Canadian J. Math., 27, 1975, 555–560.

Computations

n	Upper Bound	KMS	OSDS	BEHÓC	Computation
23	$\sqrt{45} \cdot 22^{11}$	0.3882	-	-	0.7090
27	$\sqrt{53} \cdot 26^{13}$	0.3600	-	0.3639	0.6982
31	$\sqrt{61} \cdot 30^{15}$	0.3371	0.7060	0.4354	0.7060
35	$\sqrt{69} \cdot 34^{17}$	0.3181	-	-	0.6402
39	$\sqrt{77} \cdot 38^{19}$	0.3020	-	0.3853	0.6946^{BY}
43	$\sqrt{85} \cdot 42^{21}$	0.2881	-	0.4477	0.5684^{BY}
47	$\sqrt{93} \cdot 46^{23}$	0.2760	0.7035	0.4273	0.7035
51	$\sqrt{101} \cdot 50^{25}$	0.2653	-	0.3347	0.5300^{BY}
55	$\sqrt{109} \cdot 54^{27}$	0.2557	-	0.3936	-
59	$\sqrt{117} \cdot 58^{29}$	0.2471	-	-	-
63	$\mu \cdot 63^{1/2} \cdot 62^{3} \cdot 60^{28}$	0.2878	0.8146	0.5216	-
67	$\mu \cdot 67^{1/2} \cdot 66^{3} \cdot 64^{30}$	0.2808	-	0.4296	-
71	$\mu \cdot 71^{1/2} \cdot 70^{3} \cdot 68^{32}$	0.2742	-	-	-
75	$\mu \cdot 75^{1/2} \cdot 74^{3} \cdot 72^{34}$	0.2608	-	0.4834	-
79	$\mu \cdot 79^{1/2} \cdot 78^{3} \cdot 76^{36}$	0.2623	0.7921	-	-
83	$\mu \cdot 83^{1/2} \cdot 82^{3} \cdot 80^{38}$	0.2569	-	0.3909	-
87	$\mu \cdot 87^{1/2} \cdot 86^{3} \cdot 84^{40}$	0.2517	-	0.5222	-
91	$\mu \cdot 91^{1/2} \cdot 90^{3} \cdot 88^{42}$	0.2469	-	0.5117	-
95	$\mu \cdot 95^{1/2} \cdot 94^{3} \cdot 92^{44}$	0.2424	0.7653	-	-
99	$\mu \cdot 99^{1/2} \cdot 98^{3} \cdot 96^{46}$	0.2380	-	0.4925	-

Table: Large determinants with $n \equiv 3 \mod 4$, where $\mu = \sqrt{4 \cdot 11^6 \cdot 7^{-7}}$.