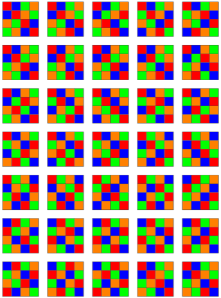
On the fractal dimension of strongly isotopism classes of Latin squares



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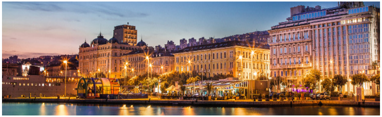

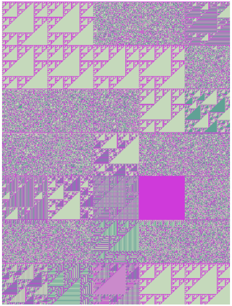

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Ruth **Moufang** (1935)

A **quasigroup of order** n is a pair $(Q,\cdot)$ formed by

- a finite set Q of n elements
- a product $\cdot$ endowed of left and right division.

Its multiplication table is a **Latin square**.

$S_n \equiv$ Symmetric group on $\{1,\ldots,n\}$.

Isotopism: $\Theta = (f,g,h) \in S_n^3$

Row-permutation (f), column-permutation (g), symbol-permutation (h).

- $f = g = h \Rightarrow$ Isomorphism.
- $f = g \Rightarrow$ Strong isotopism.

$L \equiv$

1	2	3
2	3	1
3	1	2

$\Rightarrow$ $L^{\Theta} \equiv$

2	3	1
3	1	2
1	2	3

$\Theta = ((1\ 2)(3),\ (1\ 2)(3),\ (2\ 3)(1)) \in S_3^3$

Latin squares as scramblers in Cryptography.

[V. Dimitrova V., S. Markovski, 2007] *Classification of quasigroups by image patterns*. Proc. 5th CIIT, 152–160.

Vesna **Dimitrova**

Smile **Markovski**

- A quasigroup $(Q, \cdot)$
- A plaintext $T = t_1 \ldots t_m$, with $t_i \in Q$
- A *leader symbol* $s \in Q$

Encryption: $E_s(T) = e_1 \ldots e_m$

$$e_i := \begin{cases} s \cdot t_1, & \text{if } i = 1, \\ e_{i-1} \cdot t_i, & \text{otherwise.} \end{cases}$$

$$\begin{cases} Q \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \\ T = 122333122333 \end{cases} \Rightarrow \begin{cases} E_1(T) = 123213312132 \\ E_2(T) = 231321123213 \\ E_3(T) = 312132231321 \end{cases}$$

Image patterns arising from Latin squares.

[V. Dimitrova V., S. Markovski, 2007] *Classification of quasigroups by image patterns*. Proc. 5th CIIT, 152–160.

Vesna **Dimitrova**

Smile **Markovski**

- A quasigroup $(Q, \cdot)$
- A plaintext $T = t_1 \dots t_m$
- A tuple of leader symbols $S = (s_1, \dots, s_{r-1})$

Image pattern: $\mathcal{P}_{S,T} = (p_{ij})$

$$p_{ij} := \begin{cases} t_j, & \text{if } i = 1, \\ s_{i-1} \cdot p_{i-1,1}, & \text{if } i > 1 \text{ and } j = 1, \\ p_{i,j-1} \cdot p_{i-1,j}, & \text{otherwise.} \end{cases}$$

$$\begin{cases} Q \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \\ T = 122333122333 \\ S = 123123 \end{cases} \Rightarrow \mathcal{P}_{S,T} =$$

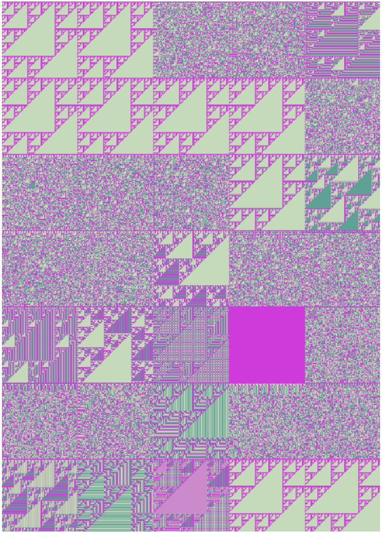
1	2	2	3	3	3	1	2	2	3	3	3
1	2	3	2	1	3	3	1	2	1	3	2
2	3	2	3	3	2	1	1	2	2	1	2
1	3	1	3	2	3	3	3	1	2	2	3
1	3	3	2	3	2	1	3	3	1	2	1
2	1	3	1	3	1	1	3	2	2	3	3
1	1	3	3	2	2	2	1	2	3	2	1

Image patterns arising from Latin squares.

$$\begin{cases} Q \equiv \begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 2 & 3 & 1 \\ \hline 3 & 1 & 2 \\ \hline \end{array} \\ T = 111111111111 \\ S = 2\ldots 2 \end{cases} \Rightarrow$$

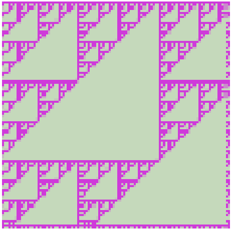
1	1	1	1	1	1	1	1	1	1	1	1
2	2	2	2	2	2	2	2	2	2	2	2
3	1	2	3	1	2	3	1	2	3	1	2
1	1	2	1	1	2	1	1	2	1	1	2
2	2	3	3	3	1	1	1	2	2	2	3
3	1	3	2	1	1	1	1	2	3	1	3
1	1	3	1	1	1	1	1	2	1	1	3
2	2	1	1	1	1	1	1	2	2	2	1
3	1	1	1	1	1	1	1	2	3	1	1
1	1	1	1	1	1	1	1	2	1	1	1
2	2	2	2	2	2	2	2	3	3	3	3
3	1	2	3	1	2	3	1	3	2	1	3
1	1	2	1	1	2	1	1	3	1	1	3
2	2	3	3	3	1	1	1	3	3	3	2
3	1	3	2	1	1	1	1	3	2	1	2
1	1	3	1	1	1	1	1	3	1	1	2
2	2	1	1	1	1	1	1	3	3	3	1
3	1	1	1	1	1	1	1	3	2	1	1
1	1	1	1	1	1	1	1	3	1	1	1
2	2	2	2	2	2	2	2	1	1	1	1
3	1	2	3	1	2	3	1	1	1	1	1
1	1	2	1	1	2	1	1	1	1	1	1
2	2	3	3	3	1	1	1	1	1	1	1
3	1	3	2	1	1	1	1	1	1	1	1
1	1	3	1	1	1	1	1	1	1	1	1
2	2	1	1	1	1	1	1	1	1	1	1
3	1	1	1	1	1	1	1	1	1	1	1
1	1	1	1	1	1	1	1	1	1	1	1

Image patterns arising from Latin squares.

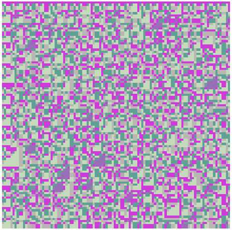


7×5 collage of image patterns arising from 35 Latin squares

Image patterns arising from Latin squares.



Fractal quasigroups
Designing error detecting codes.



Non-fractal quasigroups
Designing cryptographic primitives.

Open problem: A comprehensive analysis of their fractal dimensions.

Image patterns arising from Latin squares.

There is an interesting relation with Latin square isomorphisms:

Lemma (F., Álvarez, Gudiel, 2019)

- *Two isomorphic Latin squares L_1 and L_2 by means of isomorphism f*
- *A plaintext T*
- *A tuple of leader symbols S*

Then, $\mathcal{P}_{S,T}(L_1)$ and $\mathcal{P}_{f(S),f(T)}(L_2)$ coincide up to permutation f of their symbols.

Main question: Can we use image patterns for distinguishing non-isomorphic Latin squares?

CONTENTS

Standard sets of $r \times m$ image patterns.

- $L \in \mathcal{LS}_n$.
- $T = s \dots s$ of length m, with $s \leq n$.
- $S = (s, \dots, s)$ of length $r - 1$.

---

s-**standard** $r \times m$ **image pattern**: $\mathcal{P}_{r,m;s}(L) = \mathcal{P}_{S,T}(L)$.

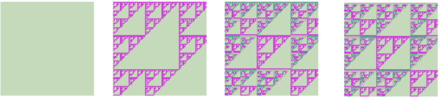
Standard set of $r \times m$ **image patterns of** L:

$$\{\mathcal{P}_{r,m;s}(L) \colon s \in \{1, \dots, n\}\}$$

$$n = 4$$
$$r = m = 90$$

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

$\Rightarrow$



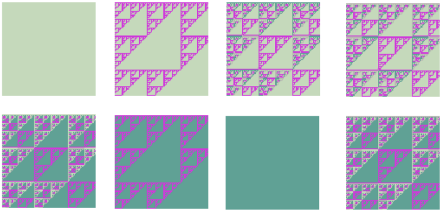
Standard sets of $r \times m$ image patterns.

Proposition

The $r \times m$ standard sets of two isomorphic Latin squares coincide, up to permutation of colors.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

3	1	4	2
4	3	2	1
1	2	3	4
2	4	1	3



Standard sets of $r \times m$ image patterns.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

$L_{4.1}$

1	2	3	4
2	1	4	3
4	3	2	1
3	4	1	2

$L_{4.2}$

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2

$L_{4.3}$

1	2	3	4
3	1	4	2
4	3	2	1
2	4	1	3

$L_{4.4}$

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

$L_{4.5}$

1	2	4	3
2	1	3	4
3	4	1	2
4	3	2	1

$L_{4.6}$

1	2	4	3
2	1	3	4
3	4	2	1
4	3	1	2

$L_{4.7}$

1	2	4	3
2	1	3	4
4	3	1	2
3	4	2	1

$L_{4.8}$

1	2	4	3
2	1	3	4
4	3	2	1
3	4	1	2

$L_{4.9}$

1	2	4	3
2	3	1	4
3	4	2	1
4	1	3	2

$L_{4.10}$

1	2	4	3
2	3	1	4
4	1	3	2
3	4	2	1

$L_{4.11}$

1	2	4	3
3	1	2	4
2	4	3	1
4	3	1	2

$L_{4.12}$

1	2	4	3
3	1	2	4
4	3	1	2
2	4	3	1

$L_{4.13}$

1	2	4	3
3	4	1	2
2	1	3	4
4	3	2	1

$L_{4.14}$

1	2	4	3
3	4	2	1
2	1	3	4
4	3	1	2

$L_{4.15}$

1	2	4	3
3	4	2	1
2	3	1	4
4	1	3	2

$L_{4.16}$

1	2	4	3
3	4	2	1
4	1	3	2
2	3	1	4

$L_{4.17}$

1	2	4	3
3	4	2	1
4	3	1	2
2	1	3	4

$L_{4.18}$

1	3	4	2
2	1	3	4
3	4	2	1
4	2	1	3

$L_{4.19}$

1	3	4	2
2	1	3	4
4	2	1	3
3	4	2	1

$L_{4.20}$

1	3	4	2
2	4	3	1
3	1	2	4
4	2	1	3

$L_{4.21}$

1	3	4	2
2	4	3	1
4	2	1	3
3	1	2	4

$L_{4.22}$

1	3	4	2
3	1	2	4
4	2	1	3
2	4	3	1

$L_{4.23}$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

$L_{4.24}$

2	1	3	4
1	3	4	2
3	4	2	1
4	2	1	3

$L_{4.25}$

2	1	3	4
1	3	4	2
4	2	1	3
3	4	2	1

$L_{4.26}$

2	1	3	4
3	4	1	2
1	2	4	3
4	3	2	1

$L_{4.27}$

2	1	3	4
3	4	2	1
1	2	4	3
4	3	1	2

$L_{4.28}$

2	1	3	4
3	4	2	1
1	3	4	2
4	2	1	3

$L_{4.29}$

2	1	3	4
3	4	2	1
4	2	1	3
1	3	4	2

$L_{4.30}$

2	1	3	4
3	4	2	1
4	3	1	2
1	2	4	3

$L_{4.31}$

2	3	1	4
1	4	2	3
3	2	4	1
4	1	3	2

$L_{4.32}$

2	3	1	4
4	1	3	2
3	2	4	1
1	4	2	3

$L_{4.33}$

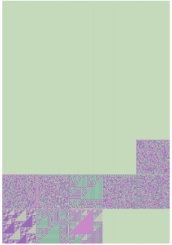
1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$L_{4.34}$

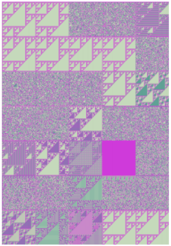
1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$L_{4.35}$

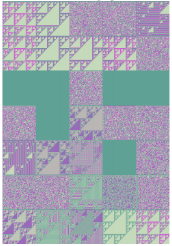
$\Rightarrow$



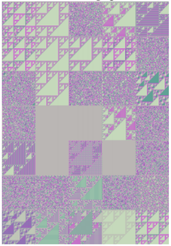
1-standard image patterns



2-standard image patterns



3-standard image patterns



4-standard image patterns

Standard sets of 90×90 image patterns arising from the

35 isomorphism classes of Latin squares of order 4.

Standard sets of $r \times m$ image patterns.

i	$\sharp cs_i$	$\sharp fs_i$	i	$\sharp cs_i$	$\sharp fs_i$	i	$\sharp cs_i$	$\sharp fs_i$
1	1	3	13	1	0	25	0	0
2	1	3	14	2	1	26	0	0
3	1	0	15	2	2	27	0	0
4	1	1	16	1	0	28	0	4
5	1	3	17	3	0	29	0	0
6	1	3	18	2	2	30	0	0
7	1	3	19	1	1	31	0	4
8	1	3	20	1	0	32	0	4
9	1	3	21	1	3	33	0	4
10	1	0	22	2	2	34	1	3
11	2	0	23	1	3	35	1	3
12	2	1	24	4	0			

Number of constant and fractal 90×90 standard image patterns of the 35 isomorphism classes of Latin squares of order 4.

Standard sets of $r \times m$ image patterns.

i	$\sharp cs_i$	$\sharp fs_i$	i	$\sharp cs_i$	$\sharp fs_i$	i	$\sharp cs_i$	$\sharp fs_i$
1	1	3	13	1	0	25	0	0
2	1	3	14	2	1	26	0	0
3	1	0	15	2	2	27	0	0
4	1	1	16	1	0	28	0	4
5	1	3	17	3	0	29	0	0
6	1	3	18	2	2	30	0	0
7	1	3	19	1	1	31	0	4
8	1	3	20	1	0	32	0	4
9	1	3	21	1	3	33	0	4
10	1	0	22	2	2	34	1	3
11	2	0	23	1	3	35	1	3
12	2	1	24	4	0			

Number of constant and fractal 90×90 standard image patterns of the 35 isomorphism classes of Latin squares of order 4.

Standard sets of $r \times m$ image patterns.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

$L_{4.1}$

1	2	3	4
2	1	4	3
4	3	2	1
3	4	1	2

$L_{4.2}$

1	2	3	4
2	3	4	1
4	1	2	3
3	4	1	2

$L_{4.3}$

1	2	3	4
3	1	4	2
4	3	2	1
2	4	1	3

$L_{4.4}$

1	2	3	4
3	4	1	2
4	3	2	1
2	1	4	3

$L_{4.5}$

1	2	4	3
2	1	3	4
3	4	1	2
4	3	2	1

$L_{4.6}$

1	2	4	3
2	1	3	4
3	4	2	1
4	3	1	2

$L_{4.7}$

1	2	4	3
2	1	3	4
4	3	1	2
3	4	2	1

$L_{4.8}$

1	2	4	3
2	1	3	4
4	3	2	1
3	4	1	2

$L_{4.9}$

1	2	4	3
2	3	1	4
3	4	2	1
4	1	3	2

$L_{4.10}$

1	2	4	3
2	3	1	4
4	1	3	2
3	4	2	1

$L_{4.11}$

1	2	4	3
3	1	2	4
2	4	3	1
4	3	1	2

$L_{4.12}$

1	2	4	3
3	1	2	4
4	3	1	2
2	4	3	1

$L_{4.13}$

1	2	4	3
3	4	1	2
2	1	3	4
4	3	2	1

$L_{4.14}$

1	2	4	3
3	4	2	1
2	1	3	4
4	3	1	2

$L_{4.15}$

1	2	4	3
3	4	2	1
2	3	1	4
4	1	3	2

$L_{4.16}$

1	2	4	3
3	4	2	1
4	1	3	2
2	3	1	4

$L_{4.17}$

1	2	4	3
3	4	2	1
4	3	1	2
2	1	3	4

$L_{4.18}$

1	3	4	2
2	1	3	4
3	4	2	1
4	2	1	3

$L_{4.19}$

1	3	4	2
2	1	3	4
4	2	1	3
3	4	2	1

$L_{4.20}$

1	3	4	2
2	4	3	1
3	1	2	4
4	2	1	3

$L_{4.21}$

1	3	4	2
2	4	3	1
4	2	1	3
3	1	2	4

$L_{4.22}$

1	3	4	2
3	1	2	4
4	2	1	3
2	4	3	1

$L_{4.23}$

1	3	4	2
4	2	1	3
2	4	3	1
3	1	2	4

$L_{4.24}$

2	1	3	4
1	3	4	2
3	4	2	1
4	2	1	3

$L_{4.25}$

2	1	3	4
1	3	4	2
4	2	1	3
3	4	2	1

$L_{4.26}$

2	1	3	4
3	4	1	2
1	2	4	3
4	3	2	1

$L_{4.27}$

2	1	3	4
3	4	2	1
1	2	4	3
4	3	1	2

$L_{4.28}$

2	1	3	4
3	4	2	1
1	3	4	2
4	2	1	3

$L_{4.29}$

2	1	3	4
3	4	2	1
4	2	1	3
1	3	4	2

$L_{4.30}$

2	1	3	4
3	4	2	1
4	3	1	2
1	2	4	3

$L_{4.31}$

2	3	1	4
1	4	2	3
3	2	4	1
4	1	3	2

$L_{4.32}$

2	3	1	4
4	1	3	2
3	2	4	1
1	4	2	3

$L_{4.33}$

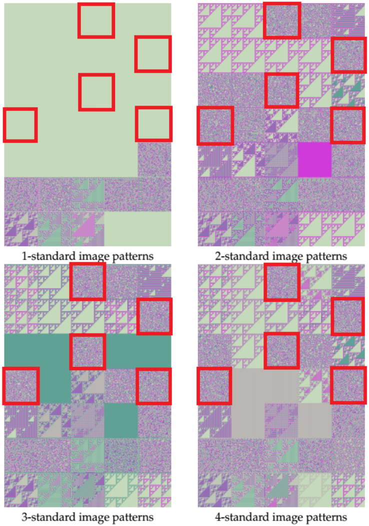
1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

$L_{4.34}$

1	2	3	4
2	1	4	3
3	4	2	1
4	3	1	2

$L_{4.35}$

$\Rightarrow$



Standard sets of 90×90 image patterns arising from the

35 isomorphism classes of Latin squares of order 4.

Standard sets of $r \times m$ image patterns.

Standard 3×3 image patterns of five distinct isomorphism classes

Can we find an efficient method for distinguishing standard sets?

CONTENTS

Homogenized standard sets.

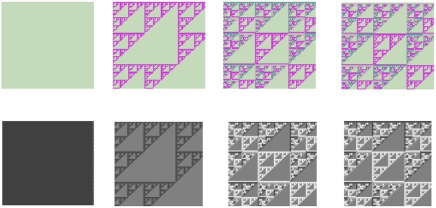
$\mathfrak{P}_n = \{c_1, \ldots, c_n\} \equiv$ Grayscale palette such that $\mathrm{Intensity}(c_i) = \frac{i}{n}$.

A standard set of image patterns of a Latin square of order n is said to be **homogenized** if the colors of $\mathfrak{P}_n$ appear in natural order (according to their intensity) when the image pixels are read row by row then column by column.

$\mathcal{H}_{r,m}(L) \equiv$ Set of homogenized standard sets of $L \in \mathrm{LS}_n$.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

$\Rightarrow$



Homogenized standard sets.

$\mathfrak{P}_n = \{c_1, \ldots, c_n\} \equiv$ Grayscale palette such that $\mathrm{Intensity}(c_i) = \frac{i}{n}$.

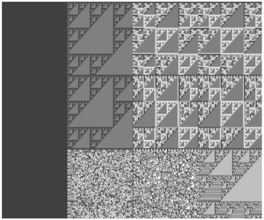
A standard set of image patterns of a Latin square of order n is said to be **homogenized** if the colors of $\mathfrak{P}_n$ appear in natural order (according to their intensity) when the image pixels are read row by row then column by column.

$\mathcal{H}_{r,m}(L) \equiv$ Set of homogenized standard sets of $L \in \mathrm{LS}_n$.

1	2	3	4
2	1	4	3
4	3	1	2
3	4	2	1

1	2	4	3
2	1	3	4
3	4	1	2
4	3	2	1

1	2	3	4
3	1	4	2
4	3	2	1
2	4	1	3



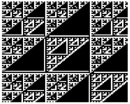
Differential box-counting fractal dimension.

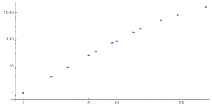
- $L \in \mathrm{LS}_n$. $\qquad \mathrm{Div}(r,m) \equiv$ Common divisors of r and m.
- $I_{i,j,k}(\mathcal{P}_{r,m;s}(L)) \equiv$ Range of gray-level intensities in the (i,j)-cell of the $\frac{r}{k} \times \frac{m}{k}$ grid of $\mathcal{P}_{r,m;s}(L)$, with $k \in \mathrm{Div}(r,m)$.

$$I_k(\mathcal{P}_{r,m;s}(L)) := \sum_{(i,j)\in\left[\frac{r}{k}\right]\times\left[\frac{m}{k}\right]} \left(1 + I_{i,j,k}(\mathcal{P}_{r,m;s}(L))\right).$$

Based on the *differential box-counting method*, we define the **differential box-counting fractal dimension** $D_B(\mathcal{P}_{r,m;s}(L))$ of $\mathcal{P}_{r,m;s}(L)$ as the slope of the linear regression line of the set of points

$$\{(\log(I_k(\mathcal{P}_{r,m;s}(L))), \log(1/k)) : \; k \in \mathrm{Div}(r,m)\}.$$





Mean fractal dimension.

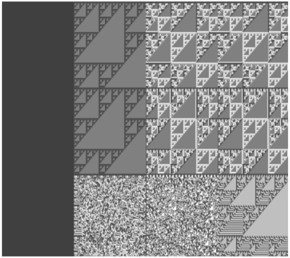
The mean value of the n differential box-counting fractal dimensions, averaged over $\mathrm{Div}(r, m)$, is the **mean fractal dimension** $D_B(\mathcal{H}_{r,m}(L))$.

Theorem

- $L_1,\ L_2 \in \mathrm{LS}_n$.

If L_1 and L_2 are isomorphic, then $D_B(\mathcal{H}_{r,m}(L_1)) = D_B(\mathcal{H}_{r,m}(L_2))$.

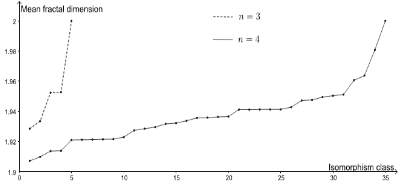
	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\ \hline 2&1&4&3\\ \hline 4&3&1&2\\ \hline 3&4&2&1\\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&4&3\\ \hline 2&1&3&4\\ \hline 3&4&1&2\\ \hline 4&3&2&1\\ \hline\end{array}$	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\ \hline 3&1&4&2\\ \hline 4&3&2&1\\ \hline 2&4&1&3\\ \hline\end{array}$
$D_B(\mathcal{P}_{90;1}(L))$	2.00000	2.00000	2.00000
$D_B(\mathcal{P}_{90;2}(L))$	1.95165	1.95165	1.92136
$D_B(\mathcal{P}_{90;3}(L))$	1.8877	1.88873	1.92331
$D_B(\mathcal{P}_{90;4}(L))$	1.8877	1.88873	1.90088
$D_B(\mathcal{H}_{90}(L))$	1.9317625	1.9322775	1.9363875

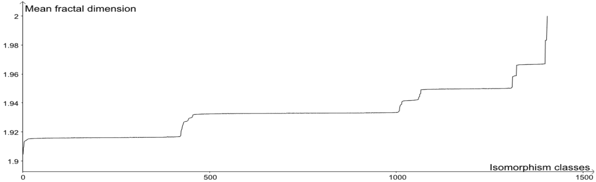


Mean fractal dimension.

$$n \in \{3,4\} \qquad (r = m = 90)$$

n	i	$D_B(n,i)$	n	i	$D_B(n,i)$	n	i	$D_B(n,i)$	n	i	$D_B(n,i)$
3	1	1.9285267	4	30	1.9212575	4	35	1.9338325	4	23	1.9428575
	5	1.9335900		29	1.9213650		19	1.9357400		15	1.9472450
	2	1.9524867		27	1.9215325		21	1.9359200		12	1.9476600
	3	1.9527467		25	1.9216950		4	1.9363875		22	1.9495350
	4	2.0000000		7	1.9230125		5	1.9366250		18	1.9504400
4	32	1.9072150		8	1.9274475		20	1.9411800		14	1.9511850
	28	1.9099250		9	1.9285825		13	1.9411950		11	1.9606500
	33	1.9137400		2	1.9296225		16	1.9413250		34	1.9637375
	31	1.9139600		1	1.9317625		10	1.9413500		17	1.9807250
	26	1.9210725		6	1.9322775		3	1.9413775		24	2.0000000



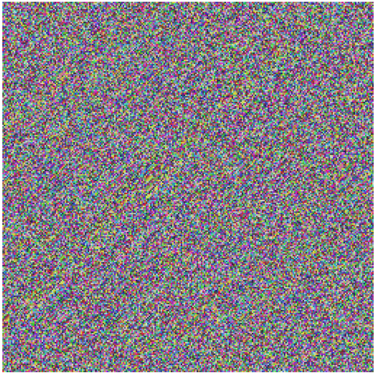


$(n = 5)$

Run time of each computation: $< 1s$ in an *Intel Core i7-8750H CPU (6 cores), with a 2.2 GHz processor and 8 GB of RAM.*

Mean fractal dimension.

$n = 256 \qquad (r = m = 90)$



Mean fractal dimension ($r = m = 90$): 1.88926

Run time: 81.63s in an *Intel Core i7-8750H CPU (6 cores), with a 2.2 GHz processor and 8 GB of RAM.*

Strong isotopism classes.

n = 3 (5 isomorphism classes)

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

1	3	2
2	1	3
3	2	1

1	3	2
3	2	1
2	1	3

2	1	3
1	3	2
3	2	1

Strong isotopism classes.

$\mathbf{n = 3}$ (5 isomorphism classes)

1	2	3
2	3	1
3	1	2

1.9285267

1	2	3
3	1	2
2	3	1

1.9524867

1	3	2
2	1	3
3	2	1

1.9527467

1	3	2
3	2	1
2	1	3

2

2	1	3
1	3	2
3	2	1

1.93359

$$\downarrow (\mathrm{Id}, \mathrm{Id}, (12))$$

2	1	3
1	3	2
3	2	1

1.93359

2	1	3
3	2	1
1	3	2

1.9527467

2	3	1
1	2	3
3	1	2

1.9524867

2	3	1
3	1	2
1	2	3

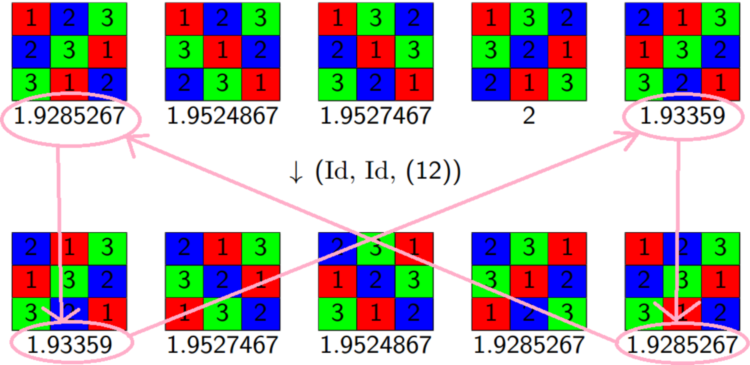
1.9285267

1	2	3
2	3	1
3	1	2

1.9285267

Strong isotopism classes.

$\underline{\mathbf{n = 3}}$ (5 isomorphism classes)



Strong isotopism classes.

n = 3 (5 isomorphism classes)

1	2	3
2	3	1
3	1	2

1.9285267
1.93359

1	2	3
3	1	2
2	3	1

1.9524867

1	3	2
2	1	3
3	2	1

1.9527467

1	3	2
3	2	1
2	1	3

2

$$\downarrow \ (\mathrm{Id},\ \mathrm{Id},\ (12))$$

2	1	3
3	2	1
1	3	2

1.9527467

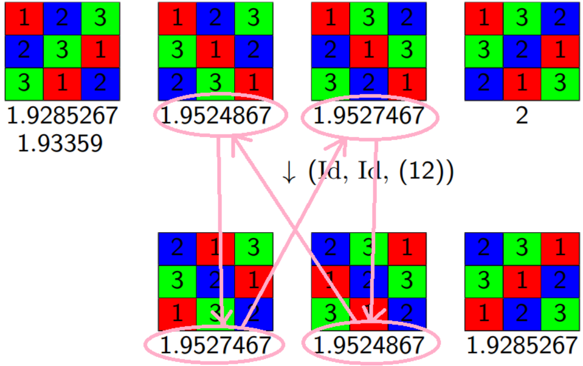
2	3	1
1	2	3
3	1	2

1.9524867

2	3	1
3	1	2
1	2	3

1.9285267

<u>n = 3</u> (5 isomorphism classes)



n = 3 (5 isomorphism classes)

1	2	3
2	3	1
3	1	2

1.9285267
1.93359

1	2	3
3	1	2
2	3	1

1.9524867
1.9527467

1	3	2
3	2	1
2	1	3

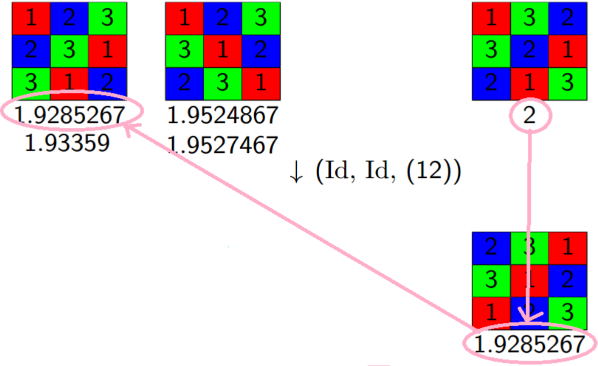
2

$\downarrow$ (Id, Id, (12))

2	3	1
3	1	2
1	2	3

1.9285267

n = 3 (5 isomorphism classes)



Strong isotopism classes.

$\underline{\mathbf{n=3}}$ (5 isomorphism classes)

1	2	3
2	3	1
3	1	2

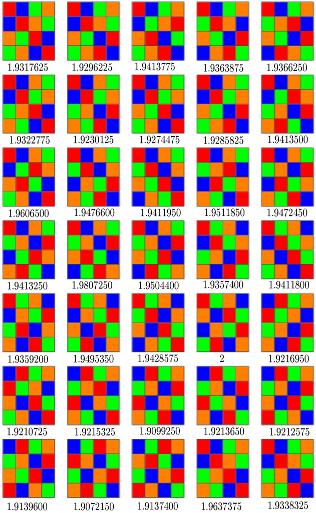
1.9285267
1.93359
2

1	2	3
3	1	2
2	3	1

1.9524867
1.9527467

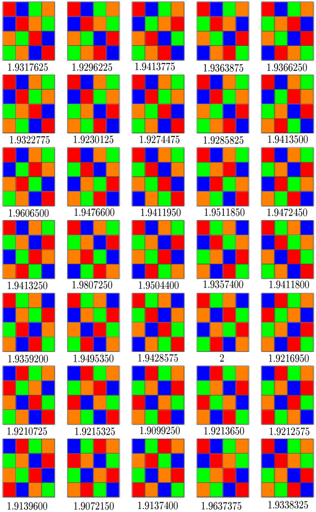
2 strong isotopism classes.

n = 4 (35 isomorphism classes)

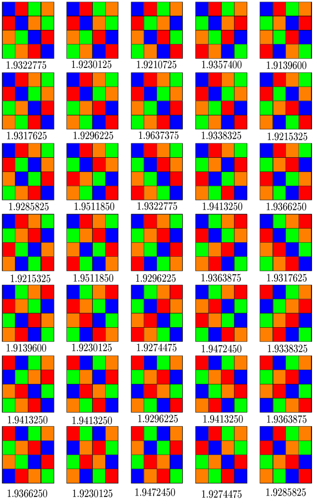


Strong isotopism classes.

n = 4 (35 isomorphism classes)

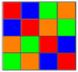


$\xrightarrow{(\mathrm{Id},\ \mathrm{Id},\ (12))}$

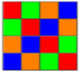


Strong isotopism classes.

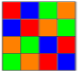
n = 4 $(4 \leq$ strong isotopism classes $\leq 8)$



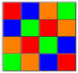
1.9317625
1.9322775
1.9411800
1.9411950



1.9274475
1.9428575
1.9637375



1.9072150
1.9099250
1.9230125
1.9296225
1.9495350
1.9504400



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1.9285825
1.9338325
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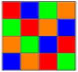

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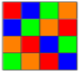

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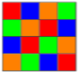
1.9413775



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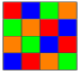
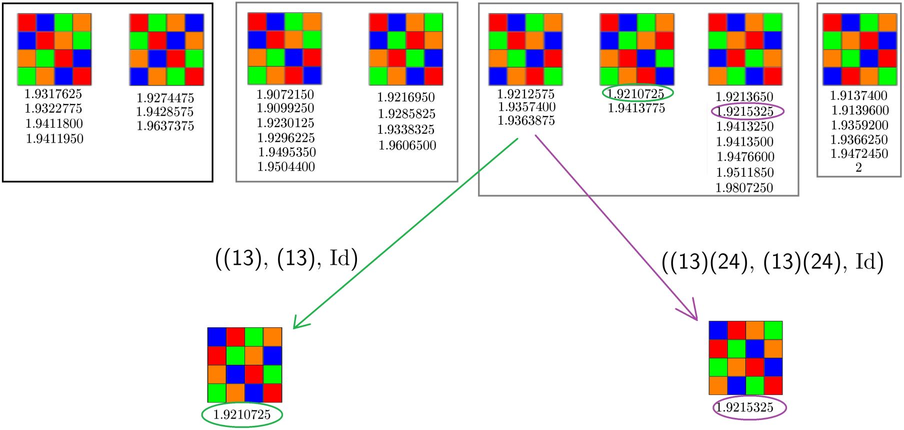
1.9215325
1.9413250
1.9413500
1.9476600
1.9511850
1.9807250



1.9137400
1.9139600
1.9359200
1.9366250
1.9472450
2

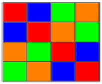
Strong isotopism classes.

n = 4 $(4 \leq$ strong isotopism classes $\leq 8)$



Strong isotopism classes.

<u>**n = 4**</u> (6 strong isotopism classes)



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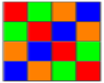
1.9411800
1.9411950



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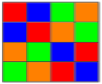
1.9428575
1.9637375



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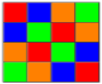

1.9216950
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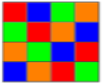

1.9210725
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1.9215325
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1.9363875
1.9413250
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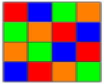
1.9511850
1.9807250



1.9137400
1.9139600
1.9359200
1.9366250
1.9472450
2

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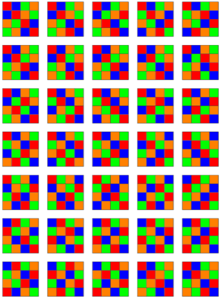
Many thanks!

On the fractal dimension of strongly isotopism classes of Latin squares



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Combinatorial Designs and Codes
Rijeka, July 12, 2021.