On some constructions of LCD codes

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Outline of the Talk

1. Preliminaries
2. Constructions of LCD codes from two-class association schemes
3. Conditions for constructing LCD codes over the field $\mathbb{F}_2$
4. LCD codes from SRGs and DRTs

Linear code

Definition

An $[\boldsymbol{n},\boldsymbol{k}]$ **linear code** $\mathcal{C}$ of length n and rank k is a k-dimensional subspace of the vector space $\mathbb{F}_q^n$, where $\mathbb{F}_q$ is the finite field with q elements.

Definition

The **Hamming distance** between two vectors $x,y \in \mathbb{F}_q^n$ is defined by

$$d(x,y) = |\{i \mid x_i \neq y_i,\ 1 \leq i \leq n\}|.$$

The **minimum distance** of a code $\mathcal{C}$ is defined by

$$d = min\{d(x,y) \mid x,y \in \mathcal{C},\ x \neq y\}.$$

An $[n,k]$ linear code with minimum distance d will be denoted by $[n,k,d]$ code.

Linear code

Definition

Given a linear $[n, k, d]$ code $\mathcal{C}$, a **generator matrix** G of $\mathcal{C}$ is a $k \times n$ matrix whose rows form a basis for a linear code.
A generator matrix of the form $G = [I_k \mid A]$, where I_k is the identity matrix of order k and A is a $k \times (n-k)$ matrix, is called a **generator matrix in standard form**.

Definition

The **dual code** of a linear code $\mathcal{C} \subset \mathbb{F}_q^n$ is the code $\mathcal{C}^\perp \subset \mathbb{F}_q^n$ where

$$\mathcal{C}^\perp = \{x \in \mathbb{F}_q^n \mid x \cdot y = 0,\ \forall y \in \mathcal{C}\}.$$

A code $\mathcal{C}$ is **self-orthogonal** if $\mathcal{C} \subseteq \mathcal{C}^\perp$ and **self-dual** if $\mathcal{C} = \mathcal{C}^\perp$.
The length n of a self-dual code is even and the dimension is $n/2$.

Strongly regular graph

Definition (SRG(v, k, λ, μ))

A simple graph G of order v is **strongly regular** with parameters (v, k, λ, μ) if

- each vertex has degree k,
- each adjacent pair of vertices has λ common neighbours,
- each nonadjacent pair of vertices has μ common neighbours.

Doubly regular tournament

Definition

A **tournament** $T = (V, E)$ of order n (**n-tournament**) is a directed graph where the vertex set V consists of n elements and the edge set $E \subset V \times V$ such that each pair of vertices x and y is joined by exactly one of the directed edges (x, y) or (y, x).

Let (x, y) be a directed edge of a tournament T. We say that x *dominates* y and y is an *out-neighbour* of x. Similarly, y *is dominated* by x and x is an *in-neighbour* of y.
The *out-degree* of the vertex x is the number of vertices that are dominated by x and the *in-degree* of the vertex x is the number of vertices that dominate x.

Doubly regular tournament

Definition

A tournament T is ***k*-regular** if each vertex dominates k vertices and is dominated by k vertices, *i.e* if every vertex in T has in-degree and out-degree k.

Definition (DRT(v, k, λ, μ))

A tournament T of order v is **doubly regular** with parameters (v, k, λ, μ) if

- T is k-regular,
- any two adjacent vertices have λ common out-neighbours,
- and each of these two vertices has additional μ out-neighbours which are not common to them.

Association scheme

Definition

Let X be a finite set of size $v \geq 2$. A **two-class association scheme** on X is a sequence of three binary relations R_0, R_1, R_2 defined on X which satisfy:

1. $X \times X = R_0 \cup R_1 \cup R_2$, $R_i \cap R_j = \emptyset$ for $i \neq j$, $i,j = 0,1,2$,
2. $R_0 = \{(x,x) \mid x \in X\}$,
3. for every $i \in \{0,1,2\}$, there exists $j \in \{0,1,2\}$ such that $R_i^T = R_j$, where $R_i^T = \{(y,x) \mid (x,y) \in R_i\}$,
4. for any triple i,j,k the number of $z \in X$ such that $(x,z) \in R_i$ and $(z,y) \in R_j$ is a constant p_{ij}^k which does not depend on the choice of x and y that satisfy $(x,y) \in R_k$.
5. $p_{ij}^k = p_{ji}^k$, for all $i,j,k \in \{0,1,2\}$.

Association scheme

The relations $R_i,\ i = 0, 1, 2$, of an association scheme can be described by their adjacency matrices $A_i,\ i = 0, 1, 2$, whose rows and columns are indexed by the elements of X and whose entries satisfy

$$(A_i)_{xy} = \begin{cases} 1 & \text{if } (x,y) \in R_i, \\ 0 & \text{otherwise.} \end{cases}$$

We have two cases:

1. $A_1^T = A_1$ and $A_2^T = A_2$ in which case the undirected graph (X, R_1) is a strongly regular graph.
2. $A_1^T = A_2$ and $A_2^T = A_1$ in which case the directed graph (X, R_1) is a doubly regular tournament.

Self-dual codes from two-class association schemes

- S. T. Dougherty, J-L. Kim, P. Solé, Double circulant codes from two-class association schemes. Adv. Math. Commun. 1 (2007), 45-64.

LCD code

Definition

A **linear code with a complementary dual** (or an **LCD code**) is a linear code $\mathcal{C}$ whose dual code $\mathcal{C}^{\perp}$ satisfies $\mathcal{C} \cap \mathcal{C}^{\perp} = \{\emptyset\}$.

If $\mathcal{C}$ is an LCD code, then $\mathcal{C}^{\perp}$ is also an LCD code.

Lemma (Massey, 1992.)

Let G be a generator matrix for a code over a field. Then $\det(GG^{\top}) \neq 0$ if and only if G generates an LCD code.

LCD codes from two-class association schemes

- D. Crnković, A. Grbac, A. Švob, Formally self-dual LCD codes from two-class association schemes, Applicable Algebra in Engineering, Communication and Computing (2021), 1-18.

Constructions of LCD codes

Let A be the adjacency matrix of a graph G with v vertices, degree k, with parameters λ and μ.

1. G is an SRG $\Rightarrow A^T = A$
2. G is a DRT $\Rightarrow A^T = \overline{A} = J - I - A$

For arbitrary scalars $r, s, t \in \mathbb{F}_q$ let $Q_{\mathbb{F}_q}(r,s,t) = (rI + sA + t\overline{A})$.

The **pure** construction is

$$P_{\mathbb{F}_q}(r,s,t) = (I \mid Q_{\mathbb{F}_q}(r,s,t)).$$

The **bordered** construction is

$$B_{\mathbb{F}_q}(r,s,t) = \left(\begin{array}{c|c|c|c} 1 & 0\ldots 0 & \alpha & \beta\ldots\beta \\ \hline 0 & & \gamma & \\ \vdots & I & \vdots & Q_{\mathbb{F}_q}(r,s,t) \\ 0 & & \gamma & \end{array}\right).$$

Pure construction

For the code $P_{\mathbb{F}_q}(r,s,t)$ to be LCD code we need $\det(P_{\mathbb{F}_q}P_{\mathbb{F}_q}^{\top}) \neq 0$.
It follows that

$$(I \mid Q_{\mathbb{F}_q}(r,s,t))(I \mid Q_{\mathbb{F}_q}(r,s,t))^T \neq \mathbf{0}.$$

We obtain that if we get

$$Q_{\mathbb{F}_q}(r,s,t)Q_{\mathbb{F}_q}(r,s,t)^T = (x-1)I\ ,\ x \neq 0,\ x \in \mathbb{F}_q.$$

Pure construction

Theorem

Let $r, s, t \in \mathbb{F}_q$ and let $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$. Further, let $P_{\mathbb{F}_q}$ be an $n \times 2n$ matrix over $\mathbb{F}_q$, and suppose $P_{\mathbb{F}_q} = \left[\ I\ \middle|\ Q_{\mathbb{F}_q}(r,s,t)\ \right]$ generates a $[2n, n]$ code C over $\mathbb{F}_q$. The code $P_{\mathbb{F}_q}(r,s,t)$ formed from an $SRG(v,k,\lambda,\mu)$, with the adjacency matrix A, is an LCD code if $x \neq 0,\ x \in \mathbb{F}_q$ and

$$\begin{aligned} r^2 + s^2k - t^2 - t^2k + t^2v &= x - 1 \\ 2rs + s^2\lambda - 2st - 2st\lambda + t^2\lambda + 2stk + t^2v - 2t^2k &= 0 \\ 2rt + s^2\mu - 2st\mu + t^2\mu + 2stk + t^2v - 2t^2 - 2t^2k &= 0\ . \end{aligned}$$

The code $P_{\mathbb{F}_q}(r,s,t)$ formed from a $DRT(v,k,\lambda,\mu)$, with the adjacency matrix A, is an LCD code if $x \neq 0,\ x \in \mathbb{F}_q$ and

$$\begin{aligned} r^2 + (s^2 + t^2)k &= x - 1 \\ rt + sr + (s^2 + t^2)(k - 1 - \lambda) + st\lambda + st\mu &= 0 \\ rt + sr + (s^2 + t^2)(k - \mu) + st\mu + st\lambda &= 0\ . \end{aligned}$$

Bordered construction

For the code $B_{\mathbb{F}_q}(r,s,t)$ to be LCD code we need $\det(B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top) \neq 0$.
We have that if $B_{\mathbb{F}_q}B_{\mathbb{F}_q}^\top$ is a diagonal matrix.
It follows that ($x \neq 0,\ y \neq 0, x, y \in \mathbb{F}_q$)

$$\begin{aligned} 1+\alpha^2+v\beta^2 &= y \\ \alpha\gamma+\beta(r+sk+t(v-k-1)) &= 0 \\ I+\gamma^2 J+Q_R(r,s,t)Q_R(r,s,t)^T &= xI. \end{aligned}$$

The third equation gives

$$Q_{\mathbb{F}_q}(r,s,t)Q_{\mathbb{F}_q}(r,s,t)^T = (x-1-\gamma^2)I - \gamma^2 A - \gamma^2\overline{A}.$$

Bordered construction

Theorem

Let $r, s, t \in \mathbb{F}_q$ *and let* $Q_{\mathbb{F}_q} = (rI + sA + t\overline{A})$*. Further, let* $B_{\mathbb{F}_q}$ *be an* $(n+1) \times (2n+2)$ *matrix over* $\mathbb{F}_q$ *and* α, β *and* γ *are scalars, and suppose*

$$B_{\mathbb{F}_q} = \left(\begin{array}{c|c|c|c} 1 & 0\dots0 & \alpha & \beta\dots\beta \\ \hline 0 & & \gamma & \\ \vdots & I & \vdots & Q_{\mathbb{F}_q}(r,s,t) \\ 0 & & \gamma & \end{array}\right)$$

generates a $[2n+2, n+1]$ *code* C *over* $\mathbb{F}_q$*.*
The code $B_{\mathbb{F}_q}(r,s,t)$ *formed from an* $SRG(v,k,\lambda,\mu)$*, with the adjacency matrix* A*, is an LCD code if* $x \neq 0,\ y \neq 0$*,* $x, y \in \mathbb{F}_q$ *and*

Bordered construction

$$\begin{aligned} r^2 + s^2k - t^2 - t^2k + t^2v &= x - 1 - \gamma^2 \\ 2rs + s^2\lambda - 2st - 2st\lambda + t^2\lambda + 2stk + t^2v - 2t^2k &= -\gamma^2 \\ 2rt + s^2\mu - 2st\mu + t^2\mu + 2stk + t^2v - 2t^2 - 2t^2k &= -\gamma^2 \\ 1 + \alpha^2 + v\beta^2 &= y \\ \alpha\gamma + \beta(r + sk + t(v - k - 1)) &= 0\ . \end{aligned}$$

The code $B_{\mathbb{F}_q}(r, s, t)$ formed from a DRT(v, k, λ, μ), with the adjacency matrix A, is an LCD code if $x \neq 0,\ y \neq 0, x, y \in \mathbb{F}_q$ and

$$\begin{aligned} r^2 + (s^2 + t^2)k &= x - 1 - \gamma^2 \\ rt + sr + (s^2 + t^2)(k - 1 - \lambda) + st\lambda + st\mu &= -\gamma^2 \\ rt + sr + (s^2 + t^2)(k - \mu) + st\mu + st\lambda &= -\gamma^2 \\ 1 + \alpha^2 + v\beta^2 &= y \\ \alpha\gamma + \beta(r + sk + t(v - k - 1)) &= 0\ . \end{aligned}$$

LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$

We simplified the preceding conditions and we gave conditions for constructing LCD codes over the fields $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$.

We constructed LCD codes from some families of strongly regular graphs and from some doubly regular tournaments.

LCD codes from SRGs over $\mathbb{F}_2$

Over $\mathbb{F}_2$, for SRGs we obtain

$$Q_{\mathbb{F}_2}(r,s,t)Q_{\mathbb{F}_2}(r,s,t)^T = (r+sk+t+tk+tv)I + (s\lambda+t\lambda+tv)A + (s\mu+t\mu+tv)\overline{A}.$$

We use this to examine when the construction gives LCD codes.

Table: Conditions for constructing LCD codes from SRGs over the field $\mathbb{F}_2$

r	s	t	Pure construction	Bordered construction
0	0	1	$v=\lambda=\mu=1+k$	$\lambda=\mu,\ \gamma=1+k+v$
0	1	0	$k=\lambda=\mu=0$	$k=\lambda=\mu=\gamma$
0	1	1	Never	Never
1	0	0	Never	Never
1	0	1	$\lambda=\mu=v=k$	$k=\lambda=\mu=v+\gamma$
1	1	0	$k=1,\ \lambda=\mu=0$	$\lambda=\mu=\gamma=k+1$
1	1	1	$v=0$	$v=\gamma$

For the bordered case, additionally it must satisfy the necessary conditions given for α and β: $\alpha+v\beta=0,\ \alpha\gamma+\beta\gamma=0$.
In the table all equalities are given in $\mathbb{F}_2$.

LCD codes from DRTs over $\mathbb{F}_2$

Over $\mathbb{F}_2$, for DRTs we obtain

$$\begin{aligned} Q_R(r,s,t)Q_R(r,s,t)^T &= (r+(s+t)k)I \\ &+ (rt+sr+(s+t)(k-1-\lambda)+st\lambda+st\mu)A \\ &+ (rt+sr+(s+t)(k-\mu)+st\mu+st\lambda)\overline{A}. \end{aligned}$$

Lemma

If Γ is a DRT with parameters (v,k,λ,μ), then $v=4\lambda+3$, $k=2\lambda+1$ and $\mu=\lambda+1$.

Table: Conditions for constructing LCD codes from DRTs over the field $\mathbb{F}_2$

r	s	t	Pure construction	Bordered construction
0	0	1	$k=0,\ \lambda=1$	$k=\gamma,\ \lambda=1$
0	1	0	$k=0,\ \lambda=1$	$k=\gamma,\ \lambda=1$
0	1	1	Never	Never
1	0	0	Never	$\gamma=1$
1	0	1	$k=\lambda=1$	$k=\gamma+1,\ \lambda=1$
1	1	0	$k=\lambda=1$	$k=\gamma+1,\ \lambda=1$
1	1	1	Never	Never

LCD codes from SRGs

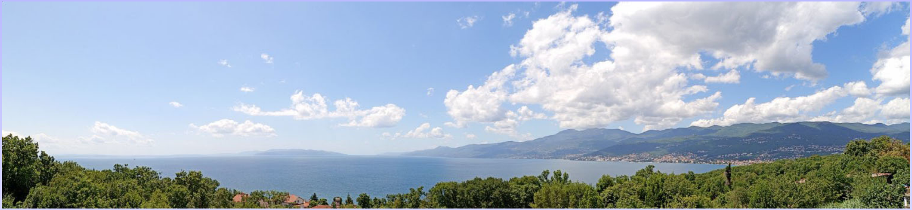
We constructed LCD codes from some families of strongly regular graphs: line graphs of complete graphs and bipartite complete graphs, some notable graphs such as the Petersen, Shrikhande, Clebsch, Hoffman-Singleton and Gewirtz graph and the Chang graphs, block graphs of Steiner triple systems, graphs obtained from orthogonal arrays and rank three permutation groups.

Graph	LCD codes	Parameters	Remark
$L(K_6)$	$B_{\mathbb{F}_2}(0,0,1)$	$[32,16,7]$	near-optimal
SRG$(15,8,4,4)$	$P_{\mathbb{F}_3}(a,0,a)$	$[30,15,8]$	
$L(K_{2,2})$	$P_{\mathbb{F}_3}(a,a,b)$, $P_{\mathbb{F}_3}(a,b,b)$	$[8,4,4]$	optimal
SRG$(4,2,0,2)$	$B_{\mathbb{F}_3}(a,a,0)$	$[10,5,4]$	near-optimal
Clebsch graph	$P_{\mathbb{F}_2}(1,0,1)$	$[32,16,8]$	optimal
SRG$(16,10,6,6)$	$B_{\mathbb{F}_2}(1,0,1)$	$[34,17,7]$	near-optimal

LCD codes from DRTs

We constructed LCD codes from DRTs of order $n = 4\lambda + 3$, $\lambda = 0, 1, \ldots, 8$.

DRT	LCD codes	Parameters	Remark
$(3,1,0,1)$	$B_{\mathbb{F}_3}(a,a,b), B_{\mathbb{F}_3}(a,b,a)$	$[8,4,4]$	optimal
	$B_{\mathbb{F}_3}(a,b,b)$	$[8,4,4]$	optimal
$(7,3,1,2)$	$P_{\mathbb{F}_3}(a,b,0), P_{\mathbb{F}_3}(a,0,b)$	$[14,7,5]$	near-optimal
	$B_{\mathbb{F}_3}(0,a,b)$	$[16,8,6]$	optimal
$(11,5,2,3)$	$B_{\mathbb{F}_3}(a,a,b), B_{\mathbb{F}_3}(a,b,a)$	$[24,12,6]$	
	$B_{\mathbb{F}_3}(0,a,0), B_{\mathbb{F}_3}(0,0,a)$	$[24,12,6]$	
$(15,7,3,4)$	$B_{\mathbb{F}_3}(a,a,b), B_{\mathbb{F}_3}(a,b,a)$	$[32,16,8]$	
	$B_{\mathbb{F}_3}(0,0,a)$	$[32,16,8]$	
$(19,9,4,5)$	$P_{\mathbb{F}_3}(a,b,0), P_{\mathbb{F}_3}(a,0,b)$	$[38,19,10]$	
	$B_{\mathbb{F}_3}(0,a,b)$	$[40,20,10]$	



Thank you for your attention!