Group Testing as Coding over the Binary Semifield via Residuation Theory

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Structure of the Presentation

The binary field $\mathbb{F}_2$

- Students of Sciences and Engineering are nowadays aware of the set $\mathbb{F}_2 = \{0, 1\}$ forming an algebraic structure known as *field*, provided we use the following operations:

$+$	0	1
0	0	1
1	1	0

$\cdot$	0	1
0	0	0
1	0	1

- Often $+$ runs under the name xor for *exclusive or* in contrast with the *inclusive or*.

- In fact, this field plays a dominant role in disciplines like Algebraic Coding Theory and Cryptography, to mention just a few.
- It gives rise to an entire powerful Linear Algebra relying on Gaussian Elimination and matrix inversion.
- Enriched with a distance function induced by the Hamming weight: $w_H : \mathbb{F}_2 \longrightarrow \mathbb{N}$ with

$$w_H(x) = \begin{cases} 1 & : \quad x \neq 0 \\ 0 & : \quad x = 0 \end{cases}$$

(and its additive extension) we enter Combinatorial Linear Algebra, another fancy term for Coding Theory.

The binary semifield $\mathbb{B}_2$

- This talk was motivated by the ongoing CoVID-19 pandemic and a mechanism that is called group testing.
- To get prepared, we will look into the *inclusive or* instead of the exclusive or xor.

$+$	0	1
0	0	1
1	1	1

$\cdot$	0	1
0	0	0
1	0	1

This semifield is at the same time the smallest non-trivial Boolean lattice, and it is natural to expect elements from order theory entering the game.

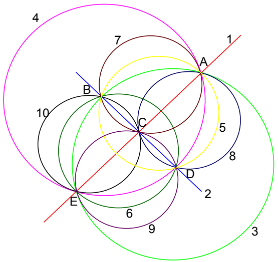
Group Testing

- Most pandemics initially spread slowly and show their true nature of growth (exponential) only at a later stage.
- At this stage the necessary counter measures have typically already become effectless.
- In the very early phase, particularly if a virus is novel, testing techniques might be complicated or costly.
- The goal is therefore to exploit testing resources efficiently.
- Around 1943, Dorfman [6] divised a technique, that is running under the name *group testing*.
- In the seventies of the previous century, a matrix-driven formalization of group testing took the lead.

Group Testing

- The underlying idea is that each test is used for a pool of specimen from different participants, while the specimen of every participant is spread over different pools.
- Dorfman observed, that it was possible to identify a small number of infected participants in a larger batch without having to medically check each individual.
- To organize group tests it is useful to have a table (a binary matrix) for tests and participants showing which participant's specimen is contained in which pool.
- We assume that we have only one round of testing, an approach known as *non-adaptive* group testing.

A small example



	1	2	3	4	5	6	7	8	9	10
A	1	0	1	1	1	0	1	1	0	0
B	0	1	1	0	1	1	1	0	0	1
C	1	1	0	0	0	0	1	1	1	1
D	0	1	0	1	1	1	0	1	1	0
E	1	0	1	1	0	1	0	0	1	1

- The table is the incidence matrix of the structure on the left side.
- This scheme allows to identify a single infected participant out of 10 by using only 5 tests.
- For more than one infected participant, the scheme will however fail.

- For example, assuming that exactly participant 3 is infected, we find that:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

- This means, that test A, B, and E will be positive, while the remaining tests will show a negative result.
- The resulting vector $[1, 1, 0, 0, 1]^T$ uniquely determines participant 3 as infected.

Goals of Group Testing

- This small scheme shows at least the principle of identifying one out of 10 with just 5 tests.
- What to do, if we know that there are 2 infected participants in a group of 20 ?
- Dividing the 20 into two batches of 10 participants is possible, but suboptimal in principle!
- **Goals of Group Testing:**
  - For a group of size n that contains up to d infected individuals, divise a scheme of k tests that allow identification of all involved infected individuals.
  - Representing the scheme by the binary $k \times n$-matrix H solve the syndrome decoding problem $Hx = s \in \mathbb{B}_2^k$.

Mathematical Modelling of Group Testing

- **Definition:** Let n, k, and d be natural numbers with $d, k \leq n$. A group testing scheme is a $k \times n$-matrix H over the semifield $\mathbb{B}_2$ satisfying the following property:

  (S) The restriction of the mapping

  $$H : \mathbb{B}_2^n \longrightarrow \mathbb{B}_2^k, \; x \mapsto Hx$$

  to the disk of Hamming radius $d - 1$ centered in the origin is an injection.

- **Remark:** Group testing schemes are the check matrices of a Coding Theory over $\mathbb{B}_2$. The test result vector takes the role of the syndrome, and the all-0-word is the information transmitted, but then distorted by the infected cases.

- We suggest to refer to H as an $[n, k, d]$ group testing scheme.
- As a matter of fact, maximizing d and minimizing k are conflicting goals.
- For given n, d with $d \leq n$ an $[n, k, d]$ group testing scheme is called *optimal*, if for every $[n, k', d]$ group testing scheme there holds $k' \geq k$.
- **Further Goals:** Construct optimal group testing schemes, and develop and implement efficient syndrome decoders for them.

Residuated Mappings

Definition

Let $(A, \leq)$ and $(B, \leq)$ be two partially ordered sets. For mappings $f : A \longrightarrow B$ and $g : B \longrightarrow A$, the pair (f, g) is called a *residuated pair*, if there holds

$$f(x) \leq y \iff x \leq g(y), \quad \text{for all } x \in A \text{ and } y \in B.$$

A few points can be easily taken from [blyth].

1: $f : A \longrightarrow B$ may be called a *residuated mapping*, if there is $g : B \longrightarrow A$, such that (f, g) is a residuated pair. The mapping g is then uniquely determined by f. Dually, f is uniquely determined by g which is called the *residual* of f. It is usually denoted by f^+.

Residuated Mappings

2: f and f^+ are monotone mappings, and there holds $f^+ \cdot f \geq \mathrm{id}_A$ and $f \cdot f^+ \leq \mathrm{id}_B$. Conversely, if two monotone mappings f and g satisfy $g \cdot f \geq \mathrm{id}_A$ and $f \cdot g \leq \mathrm{id}_B$, then they will form a residuated pair.

3: $f \cdot f^+ \cdot f = f$ and $f^+ \cdot f \cdot f^+ = f^+$, and the sets $C := \{f^+ \cdot f(x) \mid x \in A\}$ (of closed elements in A) and $K := \{f \cdot f^+(y) \mid y \in B\}$ (of kernel elements in B) indeed form a closure/kernel system in their respective spaces A and B.

4: The according closure and kernel operators on A and B are induced by $h := f^+ \cdot f$ and $k := f \cdot f^+$, respectively. The mappings $f|C$ and $f^+|K$ are mutually inverse.

A Type of Inversion

5: If A and B are complete lattices, then f is residuated if and only if $f(\sum X) = \sum f(X)$ for all $X \subseteq A$. Accordingly g is a residual mapping iff $g(\prod Y) = \prod g(Y)$ for all $Y \subseteq B$.

6: Any residuated mapping on $\mathbb{B}_2^n \longrightarrow \mathbb{B}_2^k$ can be represented by a $k \times n$ matrix with entries in $\mathbb{B}_2$. The representation of its residual mapping is the subject of the following theorem.

Theorem

Let H be a $k \times n$-matrix describing a residuated mapping $\mathbb{B}_2^n \longrightarrow \mathbb{B}_2^k$. Let N_n and N_k denote the negation on $\mathbb{B}_2^n$ and $\mathbb{B}_2^k$, respectively. Then the residual mapping of H is $H^+ : \mathbb{B}_2^k \longrightarrow \mathbb{B}_2^n,\ y \mapsto N_n H^T N_k(y)$.

A Decision Scheme

- The above theorem practically yields a decision scheme $H^+ : \mathbb{B}_2^k \longrightarrow \mathbb{B}_2^n$.
- It works in the error-free syndromes case, because all occurring syndromes in $\mathbb{B}_2^k$ are then kernel elements, as described above.
- It will be correct, if the infection pattern in $\mathbb{B}_2^n$ is a closed element.
- If this is not the case, it will return the closure of the actual infection pattern, and hence in the worst case only false positive results.
- The natural interest is therefore in residuated mappings, where the all elements up to a given Hamming weight are closed elements.

Incidence Structures

- Recall that we used $k \times n$-matrices over $\mathbb{B}_2$ in order to set up a group testing scheme.
- In our initial motivating examples, they came from geometric structures.
- We will take this to a more rigid treatment.
- **Definition:** An *incidence structure* is a pair (P, B), where P is a set of points, and B, called the set of blocks, is a subset of 2^P.
- If a point $p \in P$ is contained in the block $C \in B$, then we say that p is *incident* with C.

Incidence Structures and Partial Linear Spaces

- If (P, B) is an incidence structure with $|P| = v$ and $|B| = b$. A binary matrix $M \in \mathbb{B}_2^{v \times b}$ is called an *incidence matrix* for (P, B), if its rows are labelled by the points in P, while its columns are labelled by the blocks in B, such that

$$M_{p,C} = \begin{cases} 1 & : \quad p \in C, \\ 0 & : \quad \text{otherwise.} \end{cases}$$

- An incidence structure (P, B) is called a *partial linear space* of order (s, t) if the following axioms hold:
  - every line is incident with $s + 1$ points, and every point is incident with $t + 1$ lines,
  - two different lines can intersect in at most one point, and two different points are connected by at most one line.

Designs

- For integers $0 \leq t \leq k \leq v$, a t*-design* is a set B of k-element subsets (blocks) of a v-element set P, such that every t-element subset of P is contained in the same number λ_t of blocks of B.
- If this is the case, then B will be referred to as a t-design with parameters (v, k, λ_t).
- For $t = 2$ these designs are known as a *balanced incomplete block design*.
- Every t-design is at the same time an s-design for all $0 \leq s \leq t$. The parameter λ_s can be computed from the other parameters of the design.

Main Technical Fact

Theorem (Identifiability of Blocks)

Let B be a $t-(v,k,1)$ Steiner System, and let $C_1,\ldots C_m$ denote a collection of m distinct blocks in B. If $k>(t-1)m$, then the following hold:

(a) $|C_1\cup\cdots\cup C_m| \;\geq\; mk-(t-1)\cdot\binom{m}{2}$.

(b) *If $C\in B$ is a block with $C\subseteq C_1\cup\cdots\cup C_m$ then $C=C_j$ for some $1\leq j\leq m$ which means C is determined.*

Remark: This statement can be proved by relatively simple counting and induction.

Corollary (Injectivity of group testing matrix)

Let H be the incidence matrix of a $t-(v,k,1)$ Steiner System with b blocks. Then the restriction of the mapping

$$H : \mathbb{B}_2^b \longrightarrow \mathbb{B}_2^v, \; x \mapsto Hx$$

to the disk of Hamming radius $d-1$ centered in the origin is injective, provided $k > (t-1)d$

Remark: The preferred incidence structures taken for group testing, are those t-designs with *small* t. Partial linear spaces form a huge class of tactical configurations ($t = 1$).

Examples

- A particularly well understood class of partial linear spaces is that of the *generalized quadrangles* introduced by J. Tits.

- **Definition:** A partial linear space (P, B) of order (s, t) is called a *generalized quadrangle*, denoted by $GQ(s, t)$, if it does not contain triangles.

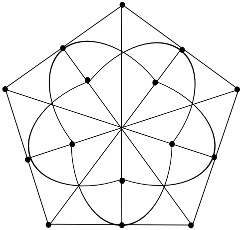


Figure: GQ(2,2)

Examples

- $GQ(s,t)$ has $(s+1)(st+1)$ points and $(t+1)(st+1)$ lines. This way, $GQ(2,4)$ has 27 points and 45 lines.
- This means (at least in theory), we obtain a group testing scheme that identifies up to 2 infected samples out of a batch of 45 using 27 tests.
- Here is the induced incidence matrix of $GQ(2,4)$.

$$\left[\begin{array}{ccccccccccccccccccccccccccccccccccccccccccccc}
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\end{array}\right]$$

False positives and false negatives

- What we have discussed so far was the error-free case.
- In the following we will consider two types of errors entering the syndrome.
- These are the so-called false positives and false negatives.
- We will of course model them by single bit flips, however note that they do not occur symmetrically.
- For the cheap antigen tests that you can purchase in the stores, it has been claimed that false negatives occur with probabilities up to 20%, while false positives are much rarer, making about 2%.

False positives and false negatives

- The above percentages describe the probabilities $P(\text{test pos} \mid \text{samp neg})$ as a false positive, and $P(\text{test neg} \mid \text{samp pos})$ as a false negative.
- For the applicant of the test, it is however much more interesting to obtain information about $P(\text{samp pos} \mid \text{test neg})$ and $P(\text{samp neg} \mid \text{test pos})$.
- These magnitudes can be related to each other which should remind us of the well-known channel forward and channel backward probabilities.
- It is known, that the distribution on the samples (here the prevalence σ as a probability) enters these relations.

False positives and false negatives

- If $P(\text{test pos} \mid \text{samp neg})$ and $P(\text{test neg} \mid \text{samp pos})$, are fixed, for example, the probability $P(\text{samp pos} \mid \text{test neg})$ will be higher under $\sigma = 90\%$ than when $\sigma = 1\%$.
- More-over, $P(\text{test pos} \mid \text{samp neg})$ assumes a possibly mixed sample, while $P(\text{samp pos} \mid \text{test neg})$ is mainly interested in the positiveness of the individual specimen.
- This means, that regarding the mentioned relationships, also the structure of the group testing scheme will play a role.

False positives and false negatives

- We have composed a little software tool in the *C* programming language in order to simulate the testing with various testing designs.
- We then fed the incidence matrices of the testing designs into the programme and ran a certain number of Monte-Carlo Simulations (typically 8192) in order to obtain an impression of the performance.
- It is clear, that sophisticated error correction mechanisms would be welcome to see the full quality of the test designs.
- In lack of such, we simply ran the *naked decoder* that I described in an earlier part of this presentation.

False positives and false negatives

- It turned out at the same time, that it makes definite sense to use as many tests as samples $k = n$, or even more of them $k \geq n$, should testing be very cheap but error-prone.
- For example, the incidence matrix of the $(7, 3, 1)$-Fano plane is of advantage when compared with the 7×7 identity matrix.
- This clearly is a (rather unexpected) extension of the original idea of group testing, where we try to minimize k under given n, because of the expensive nature of testing.
- It supports the idea of group testing just being a type of coding theory over $\mathbb{B}_2$.

Outlook

- This is work in progress. We have elaborated on the strong parallels between coding theory and group testing.
- Questions of further research may include the following:
  - Are there coding-type existence bounds for group testing? Singleten, Sphere-Packing, Elias, Varshamov, etc?
  - Optimality notions of group testing schemes depending on these existence bounds.
  - Construction of efficient decoding schemes for identification of infected individuals.
  - Are there Shannon-like theorems for the asymptotics?
  - Construction of infinite families of *asymptotically good* group testing schemes.

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Thanks for your attention!