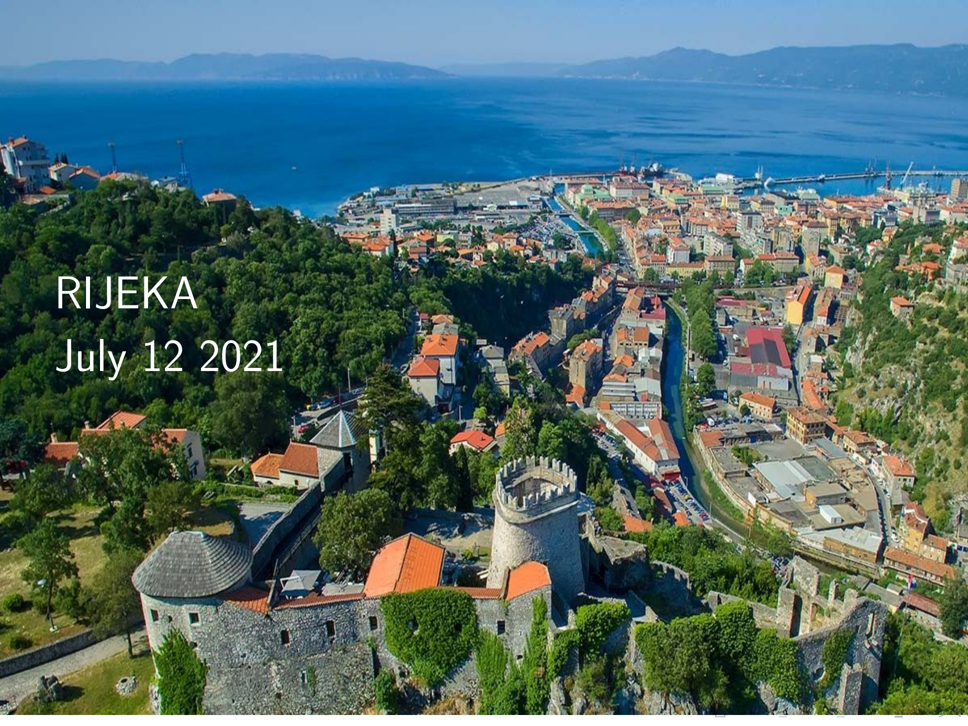
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Spectral characterizations for regular graphs

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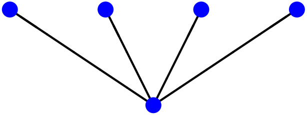
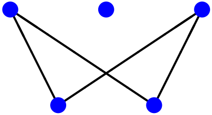
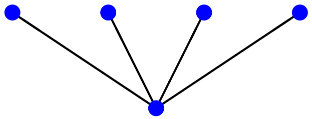
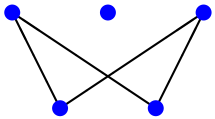




$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Cospectral with spectrum: $\{-2,\ 0,\ 0,\ 0,\ 2\}$

Properties not in common:

- being connected
- being a tree
- the girth (shortest cycle)

Theorem

Being connected, being a tree, and the girth are not characterized by the spectrum of the adjacency matrix

Common properties:

- same number of vertices, edges and triangles
- both bipartite
- both not regular

$$\lambda_1 \geq \ldots \geq \lambda_n$$

are the eigenvalues of the adjacency matrix A of G

Theorem

G has n vertices, $\frac{1}{2}\sum_{i=1}^{n} \lambda_i^2$ edges and $\frac{1}{6}\sum_{i=1}^{n} \lambda_i^3$ triangles

Theorem

G is bipartite iff $\lambda_i = -\lambda_{n+1-i}$ for $i = 1, \ldots, n$

Theorem

G is regular iff $\lambda_1 = \frac{1}{n}\sum_{i=1}^{n} \lambda_i^2$

Regular Graphs

• For regular graphs, spectral characterizations with respect to the adjacency matrix A are also valid for other types of matrices, such as
the Laplacian matrix $L = D - A$
the signless Laplacian matrix $Q = D + A$
the adjacency matrix of the complement $\overline{A} = J - A - I$

• Being regular follows from the spectrum for A, L, Q and $\overline{A}$

(D is the diagonal matrix with the degrees and
J is the all-ones matrix)

Theorem

If G is regular the following properties are characterized by the spectrum (of A, L, Q, $\overline{A}$, $S = J - 2A - I$, ...):

- the degree
- if G is bipartite
- the number of connected components
- the girth
- if G is strongly regular
- G is the incidence graph of a projective plane
- the above properties for the complement of G

Theorem

For A, L, Q and $\overline{A}$ the following properties are characterized by the spectrum:

- being regular of degree k
- being regular and bipartite
- being regular and connected
- being regular with girth g
- being strongly regular
- being the incidence graph of a projective plane
- being the complement of one of the above

Example Latin square graph

Vertices: entries of an $m \times m$ Latin square
Adjacent: same row, column, or symbol

Regular with degree $3m - 3$ and spectrum

$$\{-3^{m^2-3m+2},\ (m-3)^{3m-3},\ 3m-3\}$$

Latin square graphs of the same order have the same spectrum

a	**b**	**c**	**d**		**a**	**b**	**c**	**d**
d	**a**	**b**	**c**		**b**	**a**	**d**	**c**
c	**d**	**a**	**b**		**c**	**d**	**a**	**b**
b	**c**	**d**	**a**		**d**	**c**	**b**	**a**

Both Latin square graphs have spectrum $\{-3^6,\ 1^9,\ 9\}$
Both are regular of degree 9, not bipartite, connected, have girth 3, strongly regular

Both complements have spectrum $\{-2^9,\ 2^6,\ 6\}$
Both complements are regular of degree 6, not bipartite, connected, have girth 3, strongly regular

a	b	c	d
d	a	b	c
c	d	a	b
b	c	d	a

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

Independence number: left has 3; right has 4

Theorem

For every even $m \geq 4$ there exists a pair of Latin square graphs of order m^2 with different independence number

a	b	c	d
d	a	b	c
c	d	a	b
b	c	d	a

a	b	c	d
b	a	d	c
c	d	a	b
d	c	b	a

Chromatic number: left has 6, right has 4

Theorem

For every $m \geq 4$, $m \neq 6$ there exists a pair of Latin square graphs of order m^2 with different chromatic number

Complements give cospectral regular graphs with different clique number and clique covering number

Corollary

The independence number, the clique number, the chromatic number and the clique covering number are NOT characterized by the spectrum

Complements give cospectral regular graphs with different clique number and clique covering number

Corollary

The independence number, the clique number, the chromatic number and the clique covering number are NOT characterized by the spectrum

In general, cospectral strongly regular are a source for examples of graph properties which are not characterized by the spectrum

Many graph properties need another approach

Theorem

For the following properties there exist a pair of cospectral strongly regular graphs where one graph has the property and the other one not

- independence number
- chromatic number
- having a given automorphism group
- being a Latin square graph
- p-rank

Theorem

For the following properties there exist a pair of cospectral regular graphs where one graph has the property and the other one not

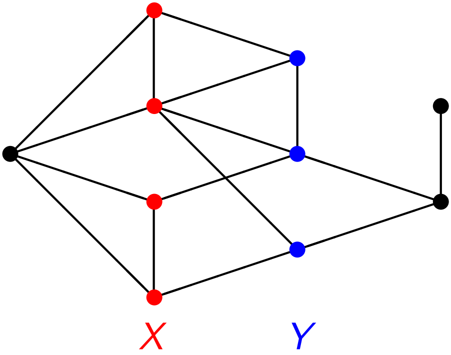
- diameter
- being distance-regular
- having a perfect matching ($\frac{n}{2}$ disjoint edges)
- vertex-connectivity
- edge-connectivity

But such a pair of strongly regular graphs cannot exist

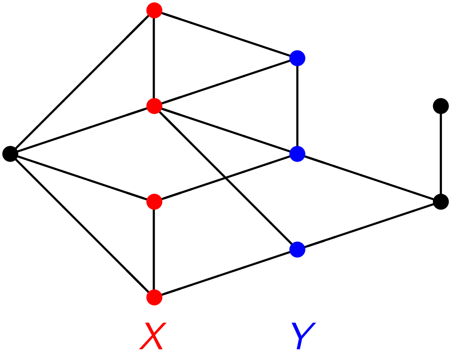
We will see such regular cospectral pairs for

- vertex-connectivity
- having a perfect matching

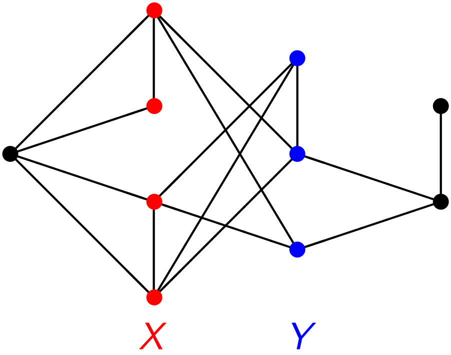
The tool is Godsil-McKay switching:



- X induces a regular graph
- A vertex in Y is adjacent to half of X
- Other vertices are adjacent to all or nothing of X



- Delete edges between X and Y
- Insert edges between X and Y that were not there



Theorem (C.D. Godsil, B.D. McKay)

Switching doesn't change the adjacency spectrum

Theorem (WHH)

For every even $k \geq 4$ there exists a pair of cospectral k-regular graph, where one has vertex-connectivity k, and the other one has vertex-connectivity $k/2 + 1$

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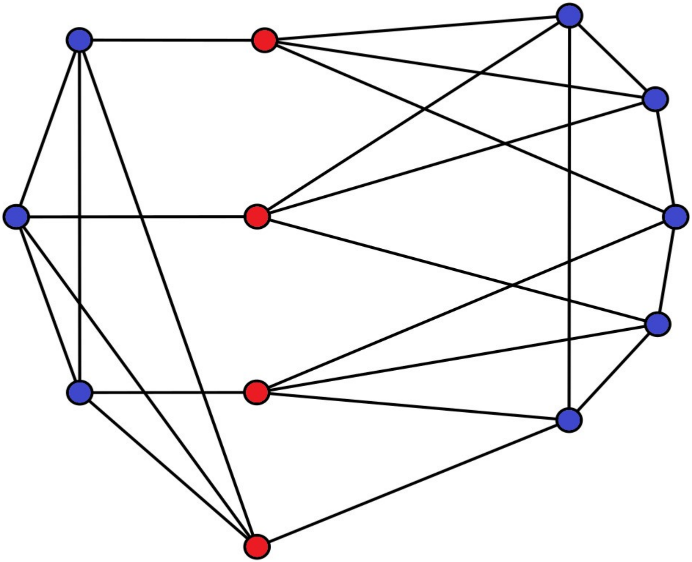
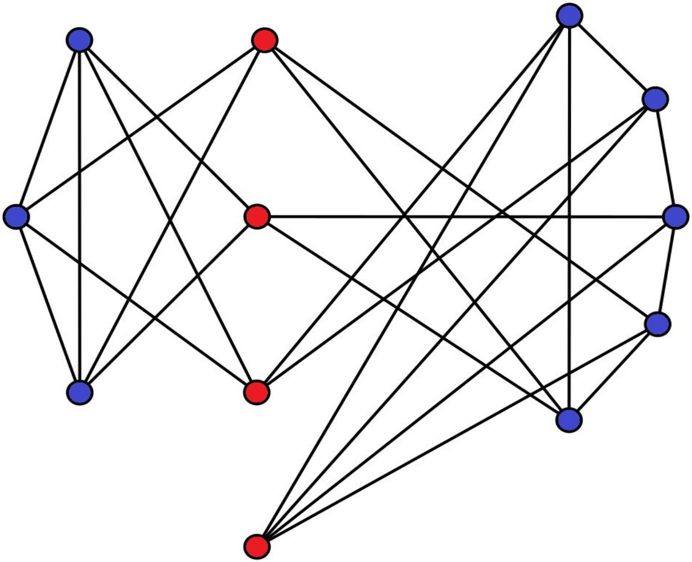
Remark Every connected strongly regular graph has vertex-connectivity k (A.E. Brouwer, D.M. Mesner)





Theorem (Z. Blázsik, J. Cummings, WHH)

For every $k \geq 5$ there exists a pair of cospectral k-regular graphs, where one has a perfect matching, and the other one not

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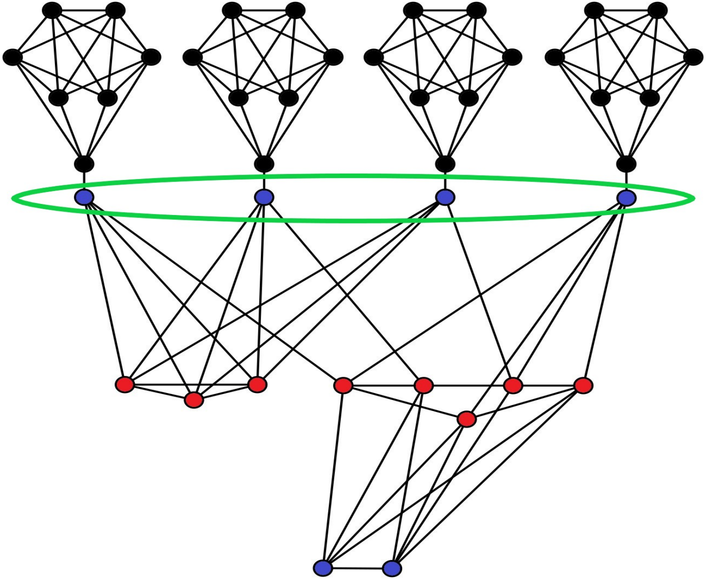
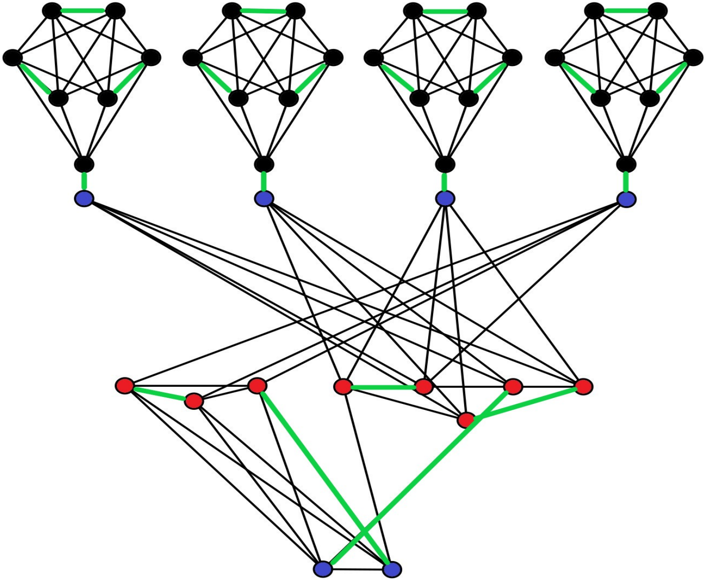
For every $k \geq 5$ there exists a pair of cospectral k-regular graphs, where one has a perfect matching, and the other one not

Remark Every connected strongly regular graph of even order has a perfect matching (A.E. Brouwer, WHH)





All mentioned graph properties characterized by the spectrum can be verified in polynomial time

The spectrum of a graph can be found in polynomial time

Question

Do there exist computationally hard graph properties which are characterized by the spectrum?

Answer (O. Etesami, WHH)

YES

Cycle representation G of a graph H with matrix

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

$$a = \begin{bmatrix} 1 & 1 & 1 & 0 & 1 & 0 \end{bmatrix}$$

$$G = C_4 + C_5 + C_6 + K_3 + C_8 + K_3$$

$$G = G_1 + \ldots + G_{\binom{n}{2}} \text{ where } G_i = \begin{cases} K_3 \text{ if } a_i = 0 \\ C_{i+3} \text{ if } a_i = 1 \end{cases}$$

Definition G has property $\mathcal{P}$ if G is the cycle representation of an Hamiltonian graph H

Theorem

1. $\mathcal{P}$ is NP-complete

2. $\mathcal{P}$ is characterized by the adjacency spectrum

Proof

1. G can be constructed from H in polynomial time
H is Hamiltonian iff G has property $\mathcal{P}$

2. Given the adjacency spectrum of G it can be checked if G is the disjoint union of cycles;
it can be verified if G is a cycle representation of an Hamiltonian graph H

Question

Are there 'more natural' NP-hard graph properties that are characterized by the spectrum?

We already saw that the independence number, the clique number, the chromatic number and the clique covering number don't work

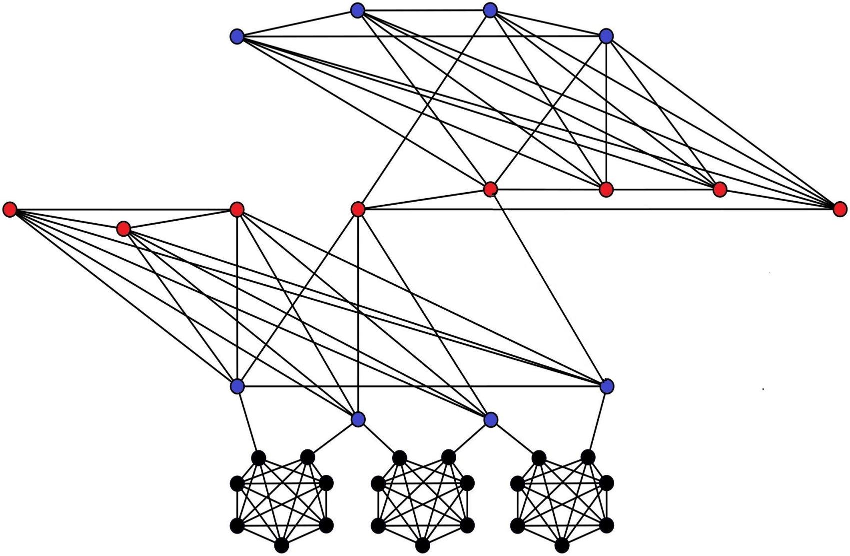
Theorem (O. Etesami, WHH)

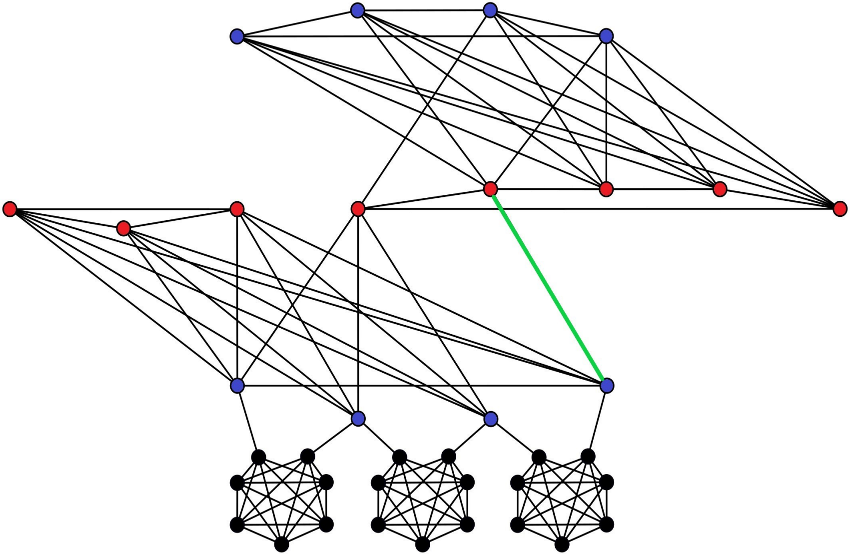
For every $k \geq 6$ there exists a pair of cospectral k-regular graphs, where one is Hamiltonian, and the other one not

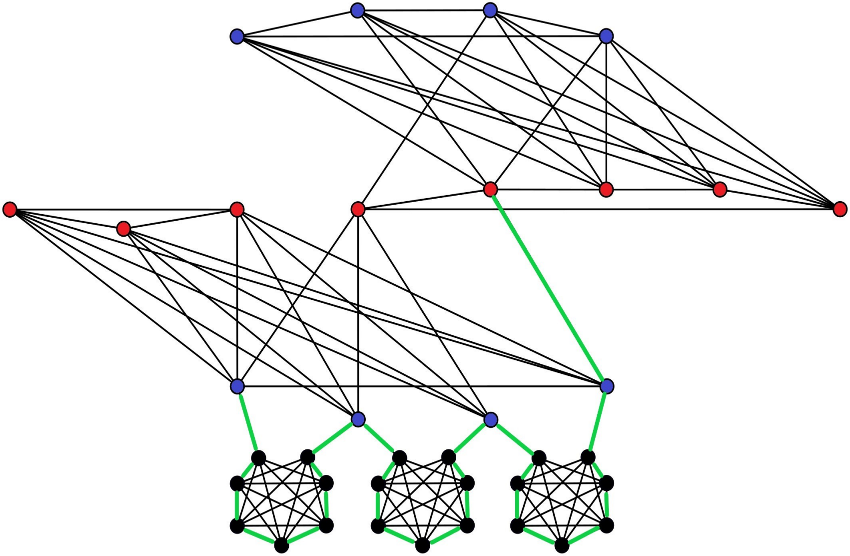
Theorem (O. Etesami, WHH)

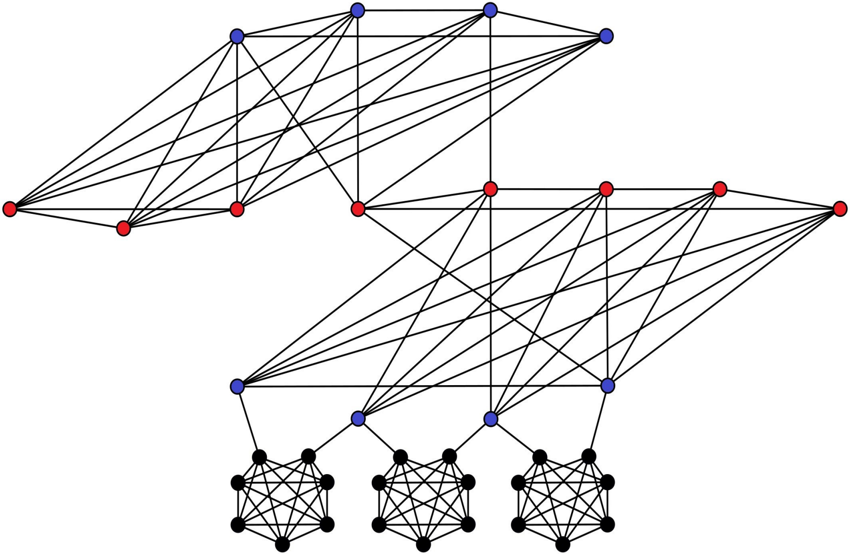
For every $k \geq 6$ there exists a pair of cospectral k-regular graphs, where one is Hamiltonian, and the other one not

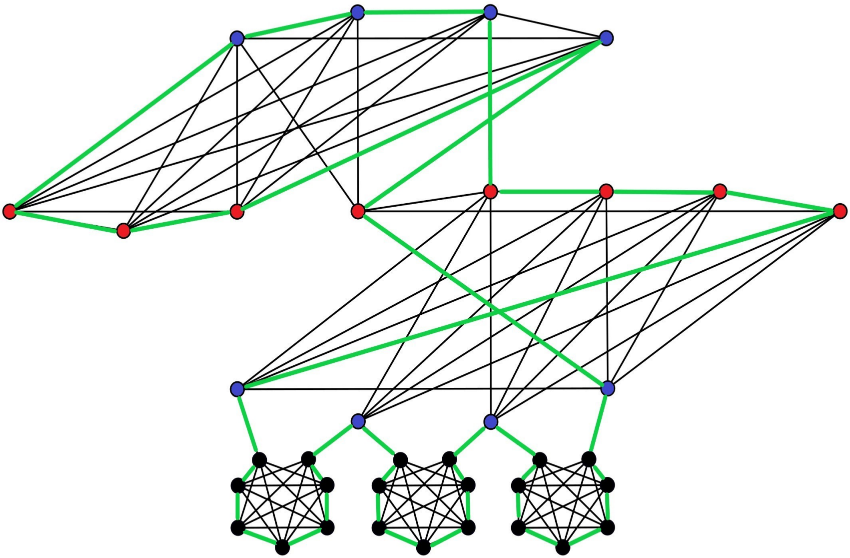
Remark Only finitely many connected strongly regular graphs are non-Hamiltonian (L. Pyber);
The Petersen graph is the only known non-Hamiltonian connected strongly regular graph



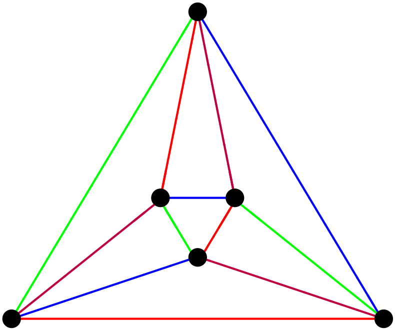








A 1-factorization of a graph is a coloring of the edges, such that each vertex meets every color exactly once.



If G has a 1-factorization with k colors, then G is regular of degree k.

For a k-regular graph the following are equivalent:

- a 1-factorization
- a partitioning of the edges into perfect matchings
- an edge coloring with k colors

Having a 1-factorization is an NP-complete property

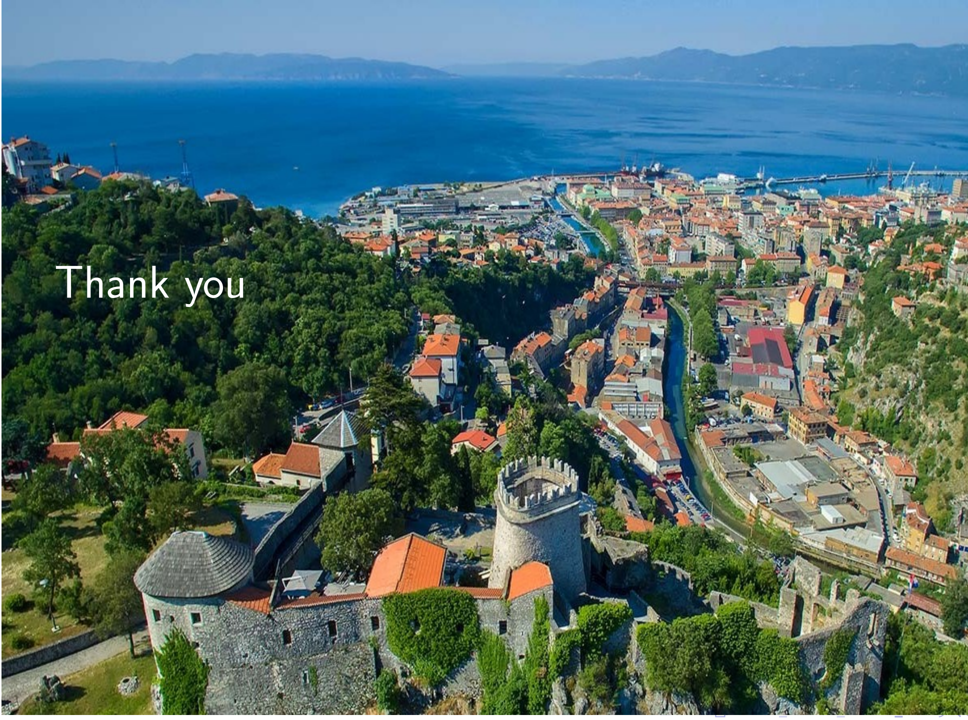
Challenge

Decide wether the property ‘having a 1-factorization’ is characterized by the spectrum

Finding a pair of cospectral graphs where one has a 1-factorization and the other one not, will be hard (if at all possible). Using switching was already nontrivial for 'perfect matching'. Also it is not likely that strongly regular graphs can be useful, because:

Theorem (S.M. Cioabă, K. Guo, WHH)
A Latin square graph of even order has a 1-factorization, and so does the complement

Conjecture
Except for the Petersen graph, a connected strongly regular graph of even order has a 1-factorization


Thank you