Neighbour-transitive codes in generalised quadrangles

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Combinatorial Designs and Codes, Rijeka (virtually)

Codes in graphs

Delsarte and Biggs independently introduced the concept of a code in a graph in 1973

- graph Γ with vertex set $V\Gamma$
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Graphs most commonly considered are the Johnson and Hamming graphs

Codes in graphs

Let C be a code in a graph Γ

Important parameters:

- *minimum distance* of C: δ
  smallest distance between distinct codewords
- *covering radius* of C: ρ
  largest distance between a vertex of Γ and its nearest codeword

s-Regular codes

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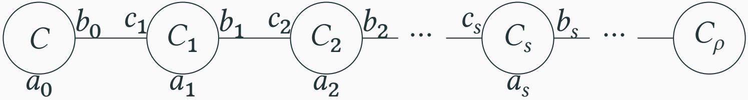


Definition (Delsarte 1973)

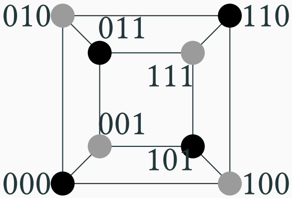
A code C is s-*regular* if for all $i \leq s$ there exist integers a_i, b_i, c_i (as above) depending only on i giving the number of adjacent vertices for each $\alpha \in C_i$.

A code that is ρ-regular is called *completely regular*.

Automorphisms Groups

Definition

The automorphism group $\mathrm{Aut}(C)$ of a code C in Γ is the set-wise stabiliser of C inside $\mathrm{Aut}(\Gamma)$.



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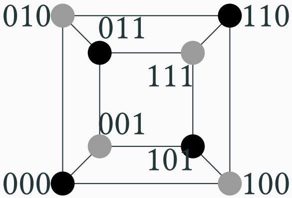


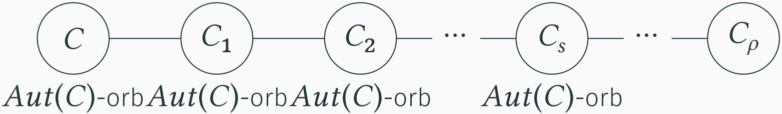
E.g. Interchanging the inner and outer pairs of vertices is an automorphism of $H(3,2)$ but not of C, whilst a reflection on a diagonal is an automorphism of both.

s-Neighbour-transitive codes

Definition

A code C is s*-neighbour-transitive* (s-NT) if $\mathrm{Aut}(C)$ acts transitively on C_i for each $i \leq s$.

If C is ρ-NT then C is said to be *completely transitive* (CT).



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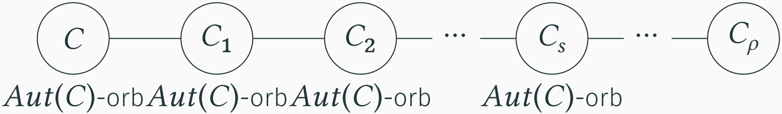


CT-ity originally introduced by Solé (1987) – above definition due to Giudici and Praeger (2000).

s-Neighbour-transitive codes

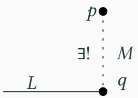
Work has been done on:

- NT codes in Johnson graphs – Liebler and Praeger, Neunhoffer and Praeger, Iopollo's PhD thesis.
- CT codes in Hamming graphs – Giudici and Praeger, Gillespie's PhD thesis, Bailey and Hawtin.
- NT codes in Hamming graphs – Gillespie's PhD thesis.
- 2-NT codes in Hamming graphs – H. thesis; Gillespie, Giudici, H. and Praeger.

Generalised quadrangles

A *generalised quadrangle* (shortly a GQ) – introduced by Tits (1959) – is an incidence structure $\mathcal{Q} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, where $\mathcal{P}$ and $\mathcal{L}$ are sets called points and lines, respectively, and I is a symmetric point-line incidence relation such that:

1. Each point is incident with $t+1$ lines ($t \geq 1$) and two distinct points are incident with at most one line.
2. Each line is incident with $s+1$ points ($s \geq 1$) and two distinct lines are incident with at most one point.
3. If p is a point and L is a line not incident with p, then there is a unique pair $(q, M) \in \mathcal{P} \times \mathcal{L}$ for which $p \mathrel{\mathrm{I}} M \mathrel{\mathrm{I}} q \mathrel{\mathrm{I}} L$.



Classical GQs

Classical GQs are associated with certain classical groups and isotropic points and lines of projective geometries under certain bilinear/sesquilinear forms.

$\mathcal{Q}$	order	$\mathcal{Q}^D$	$\mathrm{Aut}(\mathcal{Q})$
$\mathsf{Q}_4(q)$	(q,q)	$\mathsf{W}_3(q)$	$\mathrm{P}\Gamma\mathrm{O}_5(q)$
$\mathsf{Q}_5^-(q)$	(q,q^2)	$\mathsf{H}_3(q^2)$	$\mathrm{P}\Gamma\mathrm{O}_6^-(q)$
$\mathsf{W}_3(q)$	(q,q)	$\mathsf{Q}_4(q)$	$\mathrm{P}\Gamma\mathrm{Sp}_4(q)$
$\mathsf{H}_3(q^2)$	(q^2,q)	$\mathsf{Q}_5^-(q)$	$\mathrm{P}\Gamma\mathrm{U}_4(q)$
$\mathsf{H}_4(q^2)$	(q^2,q^3)	$\mathsf{H}_4(q^2)^D$	$\mathrm{P}\Gamma\mathrm{U}_5(q)$

We will be interested mainly in $\mathsf{W}_3(q)$.

Note that there are various other constructions for GQs, but we won't consider these.

Point-line incidence graph of a GQ

The point-line incidence graph Γ of a GQ $\mathcal{Q} = (\mathscr{P}, \mathscr{L}, \mathrm{I})$ is defined on the vertex set $V\Gamma = \mathscr{P} \cup \mathscr{L}$ with adjaceny given by I.

Properties of Γ:

- bipartite
- diameter 4
- girth 8

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An code C in Γ with $\delta = 4$ is either a *partial ovoid* or a *partial spread*, if $\rho = 2$ then C is either an *ovoid* or a *spread*, and if $\rho = 3$ then C is either a *maximal partial ovoid* or a *maximal partial spread*.

Ovoids and spreads

Theorem (Crnković , H., Švob)

Let C be a NT code with $\delta = 4$ and $\rho = 2$ in a thick classical GQ and assume that $\mathrm{Aut}(C)$ is insoluble. Then C is equivalent to one of the following:

1. The regular spread of $\mathsf{W}_3(q)$.
2. A classical ovoid of $\mathsf{H}_3(q^2)$.

Proof applies a theorem of Bamberg and Pentilla on transitive m-systems in polar spaces.

Examples in $\mathsf{W}_3(q)$

- $V \cong \mathbb{F}_q^4$ equipped with the symplectic form $f(x, y) = x_1 y_2 - x_2 y_1 - x_3 y_4 + x_4 y_3$
- G a sharply transitive subgroup of $\mathrm{SL}_2(q)$ (below)
- $C = \{\begin{bmatrix} I & A \end{bmatrix} \mid A \in G\}$
- note: $\det A = 1$ means form gives 0 and so the rowspaces of elements of C represent lines of $\mathsf{W}_3(q)$

q	G
2	$\mathrm{GL}_1(4)$
3	Q_8
5	$2.A_4$
7	$2.S_4$
11	$\mathrm{SL}_2(5)$

These maximal partial spreads of $\mathsf{W}_3(q)$ were originally found by Pentilla.

Main result

Theorem (Crnković , H., Švob)

Let C be a NT code with minimum distance 4 in the generalised quadrangle $\mathsf{W}_3(q)$. Then the following hold:

1. C is a regular spread $\iff \rho = 2$.
2. $|C| = q^2$ implies $\rho = 4$ and C is a spread minus a line.
3. $|C| = q^2 - 1$ implies $q = 2, 3, 5, 7$ or 11 and C is one of the codes from the previous slide, with $\rho = 3$.
4. $|C| = q + 1$ and $\rho = 3$ implies C is the set of points on a hyperbolic line.
5. If $q = 3$ and $|C| = 5$ then there is a unique code with $\rho = 3$.

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Conjecture (Crnković , H., Švob)

Let C be a NT maximal partial ovoid or maximal partial spread of $\mathrm{W}_3(q)$. Then C is as in parts 1,3,4 or 5 above.

Thanks!

Thanks for listening!