A Markov chain on the solution space of edge-colorings of bipartite graphs

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Based on joint work with István Miklós (Alfréd Rényi Institute of Mathematics)

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Preliminaries

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Fact

Any Latin rectangle can be completed into a Latin square.

A correspondence

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An *edge k-coloring* of a graph $G = (V, E)$ is a map $C : E(G) \to \{c_1, c_2, \ldots, c_k\}$ such that there is no adjacent monochromatic edges.

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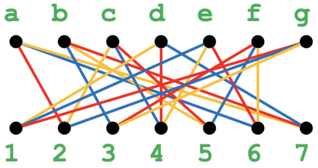
Observation

For any $(n - k) \times n$ Latin rectangle R, there is an k-regular bipartite graph G with n being both vertex class size such that completions of R and edge-k-colorings of G are in one-to-one correspondence.

Visualization

a	b	c	d	e	f	g
1	2	6	3	7	4	5
3	6	1	5	2	7	4
5	4	7	2	3	1	6
4	1	3	6	5	2	7
2	7	5	4	6	3	1
6	3	4	7	1	5	2
7	5	2	1	4	6	3



Previous results on edge colorings

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Every graph with maximum degree Δ is edge k-colorable with $k \geq \Delta + 1$ colors.

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Theorem (König, 1916)

Every bipartite graph is Δ-colorable.

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Vizing (1965), and Sanders and Zhao (2001) proved that planar graphs with $\Delta \geq 7$ are edge Δ-colorable.

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It is #P-hard to count the edge k-colorings of r-regular planar graphs for all $k \geq r \geq 3$.

It remains an open problem what is the complexity of counting 3-edge-colorings of 3-regular bipartite graphs, though the natural conjecture is that it is also #P-hard.

Metropolis-Hasting algorithm

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* The Metropolis–Hastings algorithm is a Markov chain Monte Carlo (MCMC) method that tailors a primary Markov chain to another one converging to a prescribed equilibrium distribution. In each step, a new state is proposed and accepted with a certain probability (computed from the desired equilibrium distribution and the transition probabilities of the primary Markov chain.).

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* Such a Markov chain can be efficiently used for approximate sampling if the convergence time grows only polynomially with the size of the problem instance.

Some constructed Markov chains, I

Theorem (Jacobson, Matthew, 1996)

There exists a Markov chain that converges to the uniform distribution on random Latin squares with small diameter, whose inverse of acceptance ratio under the Metropolis-Hastings algorithm is polynomially upper bounded by size of the problem instance.

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Remark

The chain constructed has "large perturbations".
That is, three not necessarily consecutive rows of a Latin square are changed at each step to get a new Latin square. Furthermore, the number of entries that might be changed in these three rows are not bounded.

Half-regular factorizations

Definition

A half-regular bipartite degree matrix $\mathcal{M} = (D, F)$ is a pair of $k \times n$ and $k \times m$ matrices of non-negative integers such that for all i, j, ℓ, $D_{i,j} = D_{i,\ell}$.
The factorization of $\mathcal{M}$ is an edge-colored bipartite graph $G(U, V, E)$ such that for any color c_i, all $u_j \in U$ and $v_\ell \in V$, the c_i color degree is $D_{i,j}$ for u_j and is $F_{i,\ell}$ for v_ℓ.

Observation

The aforementioned bijection between $(n-k) \times n$ Latin rectangle completions and edge k-colorings of a k-regular $n+n$ equibipartite graph can also be extended to half-regular factorizations of complete $k+n$ bipartite graphs with edge n-colorings.

Some constructed Markov chains, II

Theorem (Aksen, Miklós, Zhou)

There exists a Markov chain that converges to the uniform distribution on the half-regular factorizations of complete bipartite graphs with small diameter, whose inverse of acceptance ratio under the Metropolis-Hastings algorithm is polynomially upper bounded by size of the problem instance.

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Remark

This chain also uses large perturbation, i.e. it changes the color of edges incident to three vertices in one vertex class and an unbounded number of vertices in the other vertex class.

Our results, I

Theorem (H., Miklós, 2021)

Given any bipartite graph $G = (V, E)$, there exists an irreducible Markov chain $M(G, k)$ on the edge k-colorings of G such that the followings hold.

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* *The diameter of $M(G, k)$ grows linearly with $|E|$. Namely, $6|E|$ perturbations are sufficient to transform any edge coloring of G to any other one.*
* *When the Metorpolis-Hastings algorithm is applied on $M(G, k)$ to converge to the uniform distribution of edge k-colorings of G, the inverse of the acceptance ratio is upper bounded by a cubic polynomial of the number of vertices in G. Namely, the inverse of the acceptance ratio is upper bounded by $96|V|^2(|V| - 1)$.*

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* *the diameter of $M(G,k)$ is upper bounded by $3|E|$;*

Our results, II

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Theorem (H., Miklós, 2021)

Let $M(G,k)$ be the Markov chain on the edge k-colorings of a k-regular bipartite graph $G = (V,E)$. Then

- *the diameter of $M(G,k)$ is upper bounded by $3|E|$;*
- *the inverse of the acceptance ratio in the Metropolis-Hastings algorithm with the uniform distribution is upper bounded by $16|V|(|V|-1)$.*

Future direction

Conjecture (Jacobson, Mattew, 1996)

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A result when $k = 3$ would be interesting.

Thank you for your time! ☺