Constructions of divisible design Cayley graphs

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Divisible design graphs

Definition 1.

A k-regular graph on v vertices is a *divisible design graph* with parameters $(v, k, \lambda_1, \lambda_2, m, n)$ if its vertex set can be partitioned into m classes of size n, such that any two distinct vertices from the same class have λ_1 common neighbours, and any two vertices from different classes have λ_2 common neighbours.

The partition of a divisible design graph into classes is called a *canonical partition.*

Any divisible design graph Γ can be interpreted as a divisible design, by taking the vertices of Γ as points, and the neighborhoods of the vertices as blocks.

Divisible design graphs

Divisible design graphs were first introduced by three authors in a very deep and important article in 2011.

[HKM] *W.H. Haemers, H. Kharaghani, M. Meulenberg, Divisible design graphs J. Combinatorial Theory A, 118 (2011) 978–992.*

A total of twenty constructions and several conditions for existence of divisible design graphs were found.

Walk-regular and thin divisible design graphs were studied in 2014.

[CH] *D. Crnković, W. H. Haemers, Walk-regular divisible design graphs, Des. Codes Cryptogr. 72 (2014), 165–175.*

Definition 2.

Let G be a finite group with the identity element e and S be a generating set of G. If S is closed under inversion and does not contain e, then *Cayley graph* $\mathbb{C}ay(G, S)$ is a simple graph with the vertex set G in which two vertices x, y are adjacent if and only if $xy^{-1} \in S$.

There is a simple example of a divisible design Cayley graph. Let N be a proper subgroup of G. The Cayley graph $\mathbb{C}\mathrm{ay}(G, G \setminus N)$ is a complete multipartite graph whose canonical partition is the right cosets partition G by N.

Divisible design Cayley graphs

The next proposition is a special case of Theorem 1.1. from [KSh] and gives an important property of divisible design Cayley graphs.

Proposition 1.

If $\Gamma = \mathbb{C}ay(G, S)$ *is a divisible design graph, then the class of its canonical partition which contains the identity is a subgroup* N *of* G*, and the canonical partition of* Γ *coincides with the right cosets partition* G *by* N*.*

[KSh] *V.V. Kabanov, L.V. Shalaginov, On divisible design Cayley graphs, The Art of Discrete and Applied Mathematics 4 (2021) # P2.02*
Also available at http://adam-journal.eu

Construction of divisible design Cayley graphs

In this talk, we present the construction of divisible design Cayley graphs, which was published in [KSh], and give new version of this construction.

Let q be a prime power.

Let the group $\mathbb{G} = K \ltimes N$ be a semidirect product of two subgroups N and K, where $N = C_q \times \cdots \times C_q$ is the direct product of r cyclic subgroups of order q and K be a cyclic group generated by an element f^* of order $q^r - 1$ that acting without fixed points on N.

If H is a cyclic group generated by $f = (f^*)^{q-1}$, then $G = H \ltimes N$ is a group of order tq^r, where $t = |G : N| = (q^r - 1)/(q - 1)$ is the order of $H = \langle f \rangle$.

Consider the partition of G into right cosets of N and choose $\{f^i \,|\, i \in \mathbb{Z}\}$ as a set of t representatives for the cosets. Thus,

$$G = \bigcup_{i=1}^{t} N f^i.$$

Let's consider N as a linear space of dimension r over finite field $\mathbb{F}_q$.

By the formula of Gaussian binomial coefficients, there are exactly $(q^r - 1)/(q - 1)$ subgroups in N of order q^{r-1} which are hyperplanes in the linear space.

Let $\mathbb{M}$ be the set of all these subgroups in N.

Generating sets

Let's fix the cyclical ordering of $\mathbb{M} = \{M_1, M_2, \ldots, M_t\}$ as follows: $M_{i+1} = f^* M_i (f^*)^{-1}$.
Let $\varphi = [\varphi_1, \ldots, \varphi_t]$, where $\varphi_i = \varphi(i)$, be a permutation of the cyclical ordering on $\{1, \ldots, t\}$.

Moreover, let

$$f^{\varphi(i)} M_i = \{f^{\varphi(i)} a \,|\, a \in M_i\}$$

and

$$f^{\varphi(i)}(N \setminus M_i) = \{f^{\varphi(i)} a \,|\, a \in N \setminus M_i\}.$$

Generating sets

We define a generating set S_1 of G as follows:

$$S_1 = \bigcup_{i=1}^{t} f^{\varphi(i)}(N \setminus M_i).$$

[2] *V.V. Kabanov, L.V. Shalaginov, On divisible design Cayley graphs, The Art of Discrete and Applied Mathematics 4 (2021) # P2.02*

Generating sets

We define a new generating set S_2 of G as follows:

$$S_2 = (N \setminus M_1) \cup \bigcup_{i=1}^{t-1} f^{\varphi(i)}(N \setminus a_i M_i),$$

where $a_i \in N \setminus M_i$.

Theorem

Let $S \in \{S_1, S_2\}$. *If* S *is closed under inversion, then* $\Gamma = \mathbb{C}ay(G, S)$ *is a divisible design graph with parameters* $(v, k, \lambda_1, \lambda_2, m, n)$, *where*

$$v = q^r(q^r - 1)/(q-1), \quad k = q^{r-1}(q^r - 1),$$

$$\lambda_1 = q^{r-1}(q^r - q^{r-1} - 1), \quad \lambda_2 = q^{r-2}(q-1)(q^r - 1),$$

$$m = (q^r - 1)/(q-1), \quad n = q^r.$$

Moreover, Γ *has four distinct eigenvalues*

$$q^{r-1}(q^r - 1), \quad q^{t-1}, \quad 0, \quad -q^{t-1}.$$

Construction of divisible design Cayley graphs

Proposition 2.

For any $t > 2$ there is at least one permutation φ such that S is closed under inversion.

Moreover, if t is odd, then

$$\varphi = [1, t-1, t-3, \ldots, 2, t, t-2, t-4, \ldots, 3].$$

If t is even, then the permutation is more complicated and depends on the factorization of t.

Divisible design Cayley graph with small number of vertices

If $q = 2$, $r = 2$, then we have parameters $(12, 6, 2, 3, 3, 4)$.

Divisible design Cayley graph with parameters $(12, 6, 2, 3, 3, 4)$ is known as the line graph of the octahedron.

My colleagues Dmitry Panasenko and Leonid Shalaginov found all divisible design graphs up to 39 vertices by direct computer calculations. Could not find only divisible design graphs with parameters $(36, 24, 15, 16, 7, 8)$. This case turned out to be very difficult to calculate.

These results are available on the web pages

`http://alg.imm.uran.ru/dezagraphs/ddgtab.html`

If $q = 3$, $r = 2$, then we have parameters $(36, 24, 15, 16, 4, 9)$.

There are known exactly three non-isomorphic divisible design Cayley graphs with parameters $(36, 24, 15, 16, 4, 9)$. Two of them can be obtained from our Construction. We get one graph with a generating set S_1, and another with a generating set S_2.

In the tables on the next slide we show how a generating sets S_1 and S_2 is allocated to the cosets of N in G.

G := SmallGroup(36, 9);;
StructureDescription(G); "(C3 $\times$ C3) : C4"
pc-group with 4 pc-generators and relations:
$f1^2 = f2,\ f2^2 = id,\ f3^3 = id,\ f4^3 = id,$
$f3^{f1} = f3 * f4^2,\ f4^{f1} = f3^2 * f4^2,\ f3^{f2} = f3^2,\ f4^{f2} = f4^2.$
all other pairs of generators commute.
H := SylowSubgroup(G, 2);; GeneratorsOfGroup(H);; $[f1, f2]$
N := SylowSubgroup(G, 3);; GeneratorsOfGroup(N);; $[f3, f4]$

$f1^i$	$N \setminus M_i$							
f1	[	f4,		$f3*f4$,	$f4^2$,	$f3^2*f4$,	$f3*f4^2$,	$f3^2*f4^2$]
f2	[f3,	f4,	$f3^2$,		$f4^2$,	$f3^2*f4$,	$f3*f4^2$	]
f1*f2	[f3,	f4,	$f3^2$,	$f3*f4$,	$f4^2$,			$f3^2*f4^2$]
id	[f3,		$f3^2$,	$f3*f4$,		$f3^2*f4$,	$f3*f4^2$,	$f3^2*f4^2$]

M_i	[id, f3, $f3^2$]	[id, $f3*f4$, $f3^2*f4^2$]	[e, $f3^2*f4$, $f3*f4^2$]	[id, f4, $f4^2$]

$f1^i$	$N \setminus a_i M_i$								
f1	[id,	f3,			$f3*f4$,	$f4^2$,	$f3^2*f4$,		$f3^2*f4^2$]
f2	[id,			$f3^2$,	$f3*f4$,	$f4^2$,	$f3^2*f4$,	$f3*f4^2$	]
f1*f2	[id,	f3,	f4,		$f3*f4$,	$f4^2$,		$f3*f4^2$	]
id	[		f4,		$f3*f4$,	$f4^2$,	$f3^2*f4$,	$f3*f4^2$,	$f3^2*f4^2$]

a_i*M_i	$f3^2$*[id, $f3*f4$, $f3^2*f4^2$]	f3*[e, $f3^2*f4$, $f3*f4^2$]	$f3^2$*[e, f4, $f4^2$]	[e, f3, $f3^2$]

Γ_1 := CayleyGraph(G, S_1);;
AutomorphismGroup(Γ_1);;
"((((C3 x C3 x C3) : (C2 x C2)) : C3) : C2) : C2"

Γ_2 := CayleyGraph(G, S_2);;
AutomorphismGroup(Γ_2);;
"(((C3 x C3 x C3) : (C2 x C2)) : C3) : C2"

The spectra both of graphs are the same.

$$\{24^1, 3^{12}, 0^3, (-3)^{20}\}.$$

If $q = 2$, $r = 3$, then $t = 7$ and we have parameters $(56, 28, 12, 14, 7, 8)$.

There are exactly five non-isomorphic divisible design Cayley graphs with parameters $(56, 28, 12, 14, 7, 8)$. Just like in the previous case this was confirmed by direct computer calculations. They all come from a group whose structural description is (C2xC2xC2) : C7.

This is SmallGroup$(56, 11)$ in GAP.

We get three divisible design graphs by generating set S_1 and permutations $[1, 6, 4, 2, 7, 5, 3]$, $[1, 6, 3, 7, 2, 4, 5]$, $[1, 6, 2, 3, 5, 7, 4]$.

We have constructions for the two remaining graphs as well.

AutomorphismGroup(Γ_1);
"(C2 x C2 x C2 x C2 x C2) : (C7 : C3)"
AutomorphismGroup(Γ_2);
"C2 x C2 x ((C2 x C2 x C2) : C7)"
AutomorphismGroup(Γ_3);
"(C2 x C2 x C2) : (C7 : C3)"
AutomorphismGroup(Γ_4);
"(C2 x C2 x C2 x C2 x C2 x C2) : PSL(3,2)"
AutomorphismGroup(Γ_5);
"(C2 x C2 x C2) : ((C2 x C2 x C2) : PSL(3,2))"

The spectra of all these graphs are the same.

$$\{28^1, 4^{21}, 0^6, (-4)^{28}\}.$$

Calculations in GAP

```
LoadPackage( "grape" );
G := SmallGroup( 56, 11 );;
S7 := SylowSubgroup( G, 7 ); F7 := GeneratorsOfGroup( S7 );
f1 := F7[1];;
S2 := SylowSubgroup( G, 2 ); F2 := GeneratorsOfGroup( S2 );
f2 := F2[1];; f3 := F2[2];; f4 := F2[3];;
N := Group( f2, f3, f4 );; M1 := Group( f2, f3 );; M := [ ];;
for i in [ 1 .. 6 ] do
Add( M, M1^(f1^-i) );   od;
Add( M, M1 ); nel := Elements( N );; NM := [ ];;
for i in [ 1 .. 7 ] do
x := Filtered( nel, x -> not x in M[i] );
Add( NM, x ); od;
S := [ ];; phi := [ 1, 6, 4, 2, 7, 5, 3 ];
for i in phi do
for j in [ 1..4 ] do
y := ( f1^i ) * NM[Position(phi, i)][j];
Add( S, y );
od; od;
CG := CayleyGraph( G, S );
A := CollapsedAdjacencyMat( CG );
```

References

- K.T. Arasu, D. Jungnickel, A. Pott, Divisible difference sets with multiplier -1, J. Algebra 133 (1990) 35–62.
- R.C. Bose, Symmetric group divisible designs with the dual property, J. Statist. Plann. Inference 1 (1977) 87–101.
- R.C. Bose, W.S. Connor, Combinatorial properties of group divisible incomplete block designs, Ann. Math. Statist. 23 (1952) 367–383.
- T. Beth, D. Jungnickel, and H. Lenz, Design theory i, ii, 2nd ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1999.
- D. Crnković, W. H. Haemers, Walk-regular divisible design graphs, Des. Codes Cryptogr. 72 (2014), 165–175.
- M. Erickson, S. Fernando, W.H. Haemers, D. Hardy, J. Hemmeter, Deza graphs: A generalization of strongly regular graphs, J. Comb. Designs, 7 (1999) 359–405.
- S.V. Goryainov, L.V. Shalaginov, Cayley–Deza graphs with less than 60 vertices, Siberian Electronic Mathematical Reports. 11 (2014), 268–310 (In Russian)

References

- W.H. Haemers, H. Kharaghani, M. Meulenberg, Divisible design graphs J. Combinatorial Theory A, 118 (2011) 978–992.
- A. Pott, Finite geometry and character theory, Lecture Notes in Mathematics, vol. 1601, Springer-Verlag, Berlin, Heidelberg, 1995.
- A. Pott, A survey on relative difference sets, Groups, Difference Sets, and the Monster. Proceedings of a Special Research Quarter at the Ohio State University, Spring 1993 (Berlin) (K. T. Arasu, J.F. Dillon, K. Harada, S. Sehgal, and R. Solomon, eds.), Walter de Gruyter, 1996, pp. 195–232.
- A. G. Spera, Semi-regular divisible designs and Frobenius groups, Geometriae Dedicata 42 (1992) 285–294.
- R.-H. Schulz, A. G. Spera, Divisible designs and groups, Geometriae Dedicata 44 147–157, 1992.

THANK YOU VERY MUCH!