Constructing some combinatorial matrices by using orthogonal arrays

Combinatorial Designs and Codes
Satellite event of the 8th European Congress of Mathematics

Hadi Kharaghani

University of Lethbridge
Department of Mathematics and Computer Science

Joint work with Thomas Pender and Sho Suda

July 13, 2021

A weighing matrix $W(19,9)$

$$P = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
0&-&0&0&1&0&1&1&1&0&0&0&0&1&0&-&1&-&0\\
0&0&-&0&1&1&0&0&1&1&0&0&0&-&1&0&0&1&-\\
0&0&0&-&0&1&1&1&0&1&0&0&0&0&-&1&-&0&1\\
0&1&1&0&-&0&0&1&0&1&1&-&0&0&0&0&1&0&-\\
0&0&1&1&0&-&0&1&1&0&0&1&-&0&0&0&-&1&0\\
0&1&0&1&0&0&-&0&1&1&-&0&1&0&0&0&0&-&1\\
0&1&0&1&1&1&0&-&0&0&1&0&-&1&-&0&0&0&0\\
0&1&1&0&0&1&1&0&-&0&-&1&0&0&1&-&0&0&0\\
0&0&1&1&1&0&1&0&0&-&0&-&1&-&0&1&0&0&0\\
1&0&0&0&1&0&-&1&-&0&0&-&-&0&1&0&0&0&1\\
1&0&0&0&-&1&0&0&1&-&-&0&-&0&0&1&1&0&0\\
1&0&0&0&0&-&1&-&0&1&-&-&0&1&0&0&0&1&0\\
1&1&-&0&0&0&0&1&0&-&0&0&1&0&-&-&0&1&0\\
1&0&1&-&0&0&0&-&1&0&1&0&0&-&0&-&0&0&1\\
1&-&0&1&0&0&0&0&-&1&0&1&0&-&-&0&1&0&0\\
1&1&0&-&1&-&0&0&0&0&0&1&0&0&0&1&0&-&-\\
1&-&1&0&0&1&-&0&0&0&0&0&1&1&0&0&-&0&-\\
1&0&-&1&-&0&1&0&0&0&1&0&0&0&1&0&-&-&0
\end{bmatrix}$$

A twin mate to $W(19,9)$

$$Q = \begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - \\
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{bmatrix}$$

Weighing matrices

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n, k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n, n)$ is called a *Hadamard matrix*.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W($n, n-1$) is called a *conference matrix*.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.
- For $n \equiv 0 \bmod 4$
  Conjecture: There is a $W(n,k)$ for every $1 \leq k \leq n$.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.
- For $n \equiv 0 \bmod 4$
  Conjecture: There is a $W(n,k)$ for every $1 \leq k \leq n$.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.
- For $n \equiv 0 \mod 4$
  Conjecture: There is a $W(n,k)$ for every $1 \leq k \leq n$.
- For $W(n,k)$: If $n \equiv 2 \mod 4$, then $k = a^2 + b^2$, for some integer a and b.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.
- For $n \equiv 0 \bmod 4$
  Conjecture: There is a $W(n,k)$ for every $1 \leq k \leq n$.
- For $W(n,k)$: If $n \equiv 2 \bmod 4$, then $k = a^2 + b^2$, for some integer a and b.
- If n is odd, then k is a perfect square.

Weighing matrices

- A *weighing matrix* W of weight k and order n, denoted $W(n,k)$, is an $n \times n$ (-1,0,1)-matrix such that $WW^t = kI$.
- A $W(n,n)$ is called a *Hadamard matrix*.
- A W$(n, n-1)$ is called a *conference matrix*.
- For $n \equiv 0 \mod 4$
  Conjecture: There is a $W(n,k)$ for every $1 \leq k \leq n$.
- For $W(n,k)$: If $n \equiv 2 \mod 4$, then $k = a^2 + b^2$, for some integer a and b.
- If n is odd, then k is a perfect square.

Case of prime power weight

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p.

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p. By applying signed permutations, if necessary, the matrix can be written in

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right].$$

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p. By applying signed permutations, if necessary, the matrix can be written in

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right].$$

j is the column vector of p ones and 0 is the column vector of $n - p$ zeros.

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p. By applying signed permutations, if necessary, the matrix can be written in

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right].$$

j is the column vector of p ones and 0 is the column vector of $n - p$ zeros.

Lemma

For the prime power p and the positive integer m there is

an array O *in*

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p. By applying signed permutations, if necessary, the matrix can be written in

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right].$$

j is the column vector of p ones and 0 is the column vector of $n - p$ zeros.

Lemma

For the prime power p and the positive integer m there is

an array O *in p symbols of order*

$$p^{m+1} \times \left(\frac{p^{m+1} - 1}{p - 1}\right)$$

Case of prime power weight

Consider a weighing matrix of order n and weight the prime power p. By applying signed permutations, if necessary, the matrix can be written in

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right].$$

j is the column vector of p ones and 0 is the column vector of $n - p$ zeros.

Lemma

For the prime power p and the positive integer m there is

an array O in p symbols of order

$$p^{m+1} \times \left(\frac{p^{m+1} - 1}{p - 1}\right)$$

in such a way that every two different rows have exactly

$$\frac{p^m - 1}{p - 1}$$

symbols in the same column.

The use of D and R part

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

Let $\mathcal{W}$ be a $W(\frac{p^{m+1}-1}{p-1}, p^m)$. Let $\mathcal{R} = \mathcal{W} \otimes R$.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

Let $\mathcal{W}$ be a $W(\frac{p^{m+1}-1}{p-1}, p^m)$. Let $\mathcal{R} = \mathcal{W} \otimes R$.

It follows that $\mathcal{R}\mathcal{R}^t$ of order $(\frac{p^{m+1}-1}{p-1})(n-p)$ has 0 off diagonal and p^{m+1} on the diagonal.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

Let $\mathcal{W}$ be a $W(\frac{p^{m+1}-1}{p-1}, p^m)$. Let $\mathcal{R} = \mathcal{W} \otimes R$.

It follows that $\mathcal{R}\mathcal{R}^t$ of order $(\frac{p^{m+1}-1}{p-1})(n-p)$ has 0 off diagonal and p^{m+1} on the diagonal.
The matrix

$$\left[\begin{array}{c|c} \mathrm{j} & \mathcal{D} \\ \hline 0 & \mathcal{R} \end{array}\right]$$

is a weighing matrix of order $(\frac{p^{m+1}-1}{p-1})(n-1)+1$ and weight p^{m+1}.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

Let $\mathcal{W}$ be a $W(\frac{p^{m+1}-1}{p-1}, p^m)$. Let $\mathcal{R} = \mathcal{W} \otimes R$.

It follows that $\mathcal{R}\mathcal{R}^t$ of order $(\frac{p^{m+1}-1}{p-1})(n-p)$ has 0 off diagonal and p^{m+1} on the diagonal.
The matrix

$$\left[\begin{array}{c|c} \mathrm{j} & \mathcal{D} \\ \hline 0 & \mathcal{R} \end{array}\right]$$

is a weighing matrix of order $(\frac{p^{m+1}-1}{p-1})(n-1)+1$ and weight p^{m+1}.

Example

Starting with a $W(8,5)$ (a seed weighing matrix) , $n=8$, $p=5$ and $m=1$, a $W(43,25)$ is constructed.

The use of D and R part

$$W = \left[\begin{array}{c|c} \mathrm{j} & D \\ \hline 0 & R \end{array}\right]$$

By using the p rows of D as the symbols in the array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1})(n-1)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has p^{m+1} on the diagonal and -1 off diagonal.

Let $\mathcal{W}$ be a $W(\frac{p^{m+1}-1}{p-1}, p^m)$. Let $\mathcal{R} = \mathcal{W} \otimes R$.

It follows that $\mathcal{R}\mathcal{R}^t$ of order $(\frac{p^{m+1}-1}{p-1})(n-p)$ has 0 off diagonal and p^{m+1} on the diagonal.
The matrix

$$\left[\begin{array}{c|c} \mathrm{j} & \mathcal{D} \\ \hline 0 & \mathcal{R} \end{array}\right]$$

is a weighing matrix of order $(\frac{p^{m+1}-1}{p-1})(n-1)+1$ and weight p^{m+1}.

Example

Starting with a $W(8,5)$ (a seed weighing matrix) , $n=8$, $p=5$ and $m=1$, a $W(43,25)$ is constructed.

$W(43, 25)$ in details

$W(43,25)$ in details

Here is a weighing matrix $W(8,5)$:

$$H' = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array}\right].$$

$W(43, 25)$ in details

Here is a weighing matrix $W(8, 5)$:

$$H' = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array}\right].$$

Then

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 & 0 \\ - & 1 & - & 0 & 0 & 1 & 0 \\ - & - & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & - & - & - & - \end{bmatrix},$$

$W(43, 25)$ in details

Here is a weighing matrix $W(8, 5)$:

$$H' = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array}\right].$$

Then

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 & 0 \\ - & 1 & - & 0 & 0 & 1 & 0 \\ - & - & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & - & - & - & - \end{bmatrix},$$

for which $DD^t = 5I - J$.

$W(43,25)$ in details

Here is a weighing matrix $W(8,5)$:

$$H' = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array}\right].$$

Then

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 & 0 \\ - & 1 & - & 0 & 0 & 1 & 0 \\ - & - & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & - & - & - & - \end{bmatrix},$$

for which $DD^t = 5I - J$.
and

$$R = \begin{bmatrix} 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & 1 & 1 & - \end{bmatrix},$$

for which $RR^t = 5I$.

$W(43,25)$ in details

Here is a weighing matrix $W(8,5)$:

$$H' = \left[\begin{array}{c|ccccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array}\right].$$

Then

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 & 0 \\ - & 1 & - & 0 & 0 & 1 & 0 \\ - & - & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & - & - & - & - \end{bmatrix},$$

for which $DD^t = 5I - J$.
and

$$R = \begin{bmatrix} 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & 1 & 1 & - \end{bmatrix},$$

for which $RR^t = 5I$.

The Orthogonal design O

$$\begin{bmatrix} 1&1&1&1&1&1\\ 1&2&2&2&2&2\\ 1&3&3&3&3&3\\ 1&4&4&4&4&4\\ 1&5&5&5&5&5\\ 2&1&2&3&4&5\\ 2&2&3&4&5&1\\ 2&3&4&5&1&2\\ 2&4&5&1&2&3\\ 2&5&1&2&3&4\\ 3&1&3&5&2&4\\ 3&3&5&2&4&1\\ 3&5&2&4&1&3\\ 3&2&4&1&3&5\\ 3&4&1&3&5&2\\ 4&1&4&2&5&3\\ 4&4&2&5&3&1\\ 4&2&5&3&1&4\\ 4&5&3&1&4&2\\ 4&3&1&4&2&5\\ 5&1&5&4&3&2\\ 5&5&4&3&2&1\\ 5&4&3&2&1&5\\ 5&3&2&1&5&4\\ 5&2&1&5&4&3 \end{bmatrix}$$

Constructing $W(43, 25)$

Constructing W(43, 25)

Replacing the five numbers in the array O with the five rows of R (in any order) a matrix $\mathcal{D}$ is obtained for which $\mathcal{D}\mathcal{D}^t = 25 - J$.

Constructing W(43, 25)

Replacing the five numbers in the array O with the five rows of R (in any order) a matrix $\mathcal{D}$ is obtained for which $\mathcal{D}\mathcal{D}^t = 25 - J$.
If W is any W(6, 5), then

$$\mathcal{R} = W \otimes R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{bmatrix} \otimes R = \begin{bmatrix} 0 & R & R & R & R & R \\ R & 0 & R & \bar{R} & \bar{R} & R \\ R & R & 0 & R & \bar{R} & \bar{R} \\ R & \bar{R} & R & 0 & R & \bar{R} \\ R & \bar{R} & \bar{R} & R & 0 & R \\ R & R & \bar{R} & \bar{R} & R & 0 \end{bmatrix}$$

and $\mathcal{R}\mathcal{R}^t = 25I$. Moreover, $\mathcal{D}\mathcal{R}^t = 0$.

Constructing W(43, 25)

Replacing the five numbers in the array O with the five rows of R (in any order) a matrix $\mathcal{D}$ is obtained for which $\mathcal{D}\mathcal{D}^t = 25 - J$.
If W is any W(6, 5), then

$$\mathcal{R} = W \otimes R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{bmatrix} \otimes R = \begin{bmatrix} 0 & R & R & R & R & R \\ R & 0 & R & \bar{R} & \bar{R} & R \\ R & R & 0 & R & \bar{R} & \bar{R} \\ R & \bar{R} & R & 0 & R & \bar{R} \\ R & \bar{R} & \bar{R} & R & 0 & R \\ R & R & \bar{R} & \bar{R} & R & 0 \end{bmatrix}$$

and $\mathcal{R}\mathcal{R}^t = 25I$. Moreover, $\mathcal{D}\mathcal{R}^t = 0$. Consequently, the matrix

$$X = \left[\begin{array}{c|c} \mathrm{j} & \mathcal{D} \\ \hline 0 & \mathcal{R} \end{array}\right]$$

forms a W(43, 25).

Constructing W(43, 25)

Replacing the five numbers in the array O with the five rows of R (in any order) a matrix $\mathcal{D}$ is obtained for which $\mathcal{D}\mathcal{D}^t = 25 - J$.
If W is any W(6, 5), then

$$\mathcal{R} = W \otimes R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{bmatrix} \otimes R = \begin{bmatrix} 0 & R & R & R & R & R \\ R & 0 & R & \bar{R} & \bar{R} & R \\ R & R & 0 & R & \bar{R} & \bar{R} \\ R & \bar{R} & R & 0 & R & \bar{R} \\ R & \bar{R} & \bar{R} & R & 0 & R \\ R & R & \bar{R} & \bar{R} & R & 0 \end{bmatrix}$$

and $\mathcal{R}\mathcal{R}^t = 25I$. Moreover, $\mathcal{D}\mathcal{R}^t = 0$. Consequently, the matrix

$$X = \left[\begin{array}{c|c} \mathrm{j} & \mathcal{D} \\ \hline 0 & \mathcal{R} \end{array}\right]$$

forms a W(43, 25).

Balanced weighing matrices, case of $n = p + 1$

Balanced weighing matrices, case of $n = p + 1$

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

NSERC
CRSNG

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a $\mathrm{W}(v, k)$, and let $A = |W| = [|w_{ij}|]$.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a W(v, k), and let $A = |W| = [|w_{ij}|]$.If A is the incidence matrix of a symmetric $BIBD(v, k, \lambda)$, then W is called to be *balanced* and is shown as $BW(v, k, \lambda)$.

NSERC CRSNG

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a W(v, k), and let $A = |W| = [|w_{ij}|]$.If A is the incidence matrix of a symmetric $BIBD(v, k, \lambda)$, then W is called to be *balanced* and is shown as $BW(v, k, \lambda)$.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a W(v, k), and let $A = |W| = [|w_{ij}|]$.If A is the incidence matrix of a symmetric $BIBD(v, k, \lambda)$, then W is called to be *balanced* and is shown as $BW(v, k, \lambda)$.

The only known class of balanced weighing matrices is the following:

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a $\mathrm{W}(v, k)$, and let $A = |W| = [|w_{ij}|]$.If A is the incidence matrix of a symmetric $BIBD(v, k, \lambda)$, then W is called to be *balanced* and is shown as $BW(v, k, \lambda)$.

The only known class of balanced weighing matrices is the following:

Theorem

For each odd prime power p and positive integer m there is a balanced weighing matrix (with classical parameters)

$$BW\left(\frac{p^{m+1}-1}{p-1}, p^m, p^{m-1}(p-1)\right).$$

There is also a *sporadic* case of $BW(19, 9, 4)$ known.

Balanced weighing matrices, case of $n = p + 1$

- It is natural to ask if a weighing matrix exhibits any further structure.
- An important question is whether or not by squaring the entries of a weighing matrix the incidence matrix of a symmetric BIBD is obtained.

Definition

Let $W = [w_{ij}]$ be a W(v, k), and let $A = |W| = [|w_{ij}|]$.If A is the incidence matrix of a symmetric $BIBD(v, k, \lambda)$, then W is called to be *balanced* and is shown as $BW(v, k, \lambda)$.

The only known class of balanced weighing matrices is the following:

Theorem

For each odd prime power p and positive integer m there is a balanced weighing matrix (with classical parameters)

$$BW\left(\frac{p^{m+1}-1}{p-1}, p^m, p^{m-1}(p-1)\right).$$

There is also a *sporadic* case of $BW(19, 9, 4)$ known.

Proof: the Case of $n = p + 1$

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p.

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

By using the p rows of D as the symbols of the orthogonal array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1}p)$ is obtained.

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p+1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

By using the p rows of D as the symbols of the orthogonal array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1}p)$ is obtained.

It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has $p^{m+1} - 1$ on the diagonal and -1 off diagonal.

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

By using the p rows of D as the symbols of the orthogonal array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1}p)$ is obtained.
It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has $p^{m+1} - 1$ on the diagonal and -1 off diagonal.
Let $\mathcal{R} = W \otimes (\mathrm{j})^t$. Then $\mathcal{R}$ is a $\frac{p^{m+2}-1}{p-1} \times (\frac{p^{m+1}-1}{p-1}p)$ matrix and $\mathcal{R}\mathcal{R}^t$ has p^{m+1} on the diagonal and 0 off-diagonal. The matrix

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

By using the p rows of D as the symbols of the orthogonal array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1}p)$ is obtained.

It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has $p^{m+1} - 1$ on the diagonal and -1 off diagonal.

Let $\mathcal{R} = W \otimes (\mathrm{j})^t$. Then $\mathcal{R}$ is a $\frac{p^{m+2}-1}{p-1} \times (\frac{p^{m+1}-1}{p-1}p)$ matrix and $\mathcal{R}\mathcal{R}^t$ has p^{m+1} on the diagonal and 0 off-diagonal. The matrix

$$\left[\begin{array}{c|c} 0 & \mathcal{R} \\ \hline \mathrm{j} & \mathcal{D} \end{array}\right]$$

is the desired matrix.

Proof: the Case of $n = p + 1$

Let p be an odd prime power. Let W be a seed weighing matrix of order $p + 1$ with weight p. Let

$$W = \left[\begin{array}{c|c} 0 & (\mathrm{j})^t \\ \hline \mathrm{j} & D \end{array}\right]$$

By using the p rows of D as the symbols of the orthogonal array O a matrix $\mathcal{D}$ of order $p^{m+1} \times (\frac{p^{m+1}-1}{p-1}p)$ is obtained.

It follows that the matrix $\mathcal{D}\mathcal{D}^t$ (of order p^{m+1}) has $p^{m+1} - 1$ on the diagonal and -1 off diagonal.

Let $\mathcal{R} = W \otimes (\mathrm{j})^t$. Then $\mathcal{R}$ is a $\frac{p^{m+2}-1}{p-1} \times (\frac{p^{m+1}-1}{p-1}p)$ matrix and $\mathcal{R}\mathcal{R}^t$ has p^{m+1} on the diagonal and 0 off-diagonal. The matrix

$$\left[\begin{array}{c|c} 0 & \mathcal{R} \\ \hline \mathrm{j} & \mathcal{D} \end{array}\right]$$

is the desired matrix.

The sporadic case of $BW(19, 9, 4)$

The sporadic case of $BW(19, 9, 4)$

Here it is:

The sporadic case of $BW(19, 9, 4)$

Here it is:

$$
\left[\begin{array}{c|cccccccccccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{array}\right]
$$

$BW(19, 9, 4)$

The sporadic case of $BW(19,9,4)$

Here it is:

$$
\left[\begin{array}{c|cccccccccccccccccc}
1 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
1 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
1 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
1 & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 \\
\hline
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{array}\right]
$$

$BW(19,9,4)$

Blowing $BW(19,9,4)$ to an infinite class

Blowing $BW(19, 9, 4)$ to an infinite class

There is a

$$BW\left(1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m\right)$$

Blowing $BW(19, 9, 4)$ to an infinite class

There is a

$$BW\left(1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m\right)$$

for every nonnegative integer m.

In order to construct the infinite class we need to show

- a method to construct an infinite number of symmetric designs from a single seed symmetric design.
- a method to *sign* the symmetric design.

Quasi-residual designs

Lemma

Let $(p+1, 2p, p, \frac{p+1}{2}, \frac{p-1}{2})$ *be parameters of a quasi-residual BIBD, say* D_1.

Quasi-residual designs

Lemma

Let $(p+1, 2p, p, \frac{p+1}{2}, \frac{p-1}{2})$ be parameters of a quasi-residual BIBD, say D_1. Let W be a weighing matrix $W(p+1, p)$. Then there is a quasi-residual

$$BIBD((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})).$$

Proof: Let D_2 be the *mate* residual design to D_1, then $D_1 + D_2 = J_{(p+1)\times 2p}$.

Quasi-residual designs

Lemma

Let $(p+1, 2p, p, \frac{p+1}{2}, \frac{p-1}{2})$ *be parameters of a quasi-residual BIBD, say* D_1*. Let* W *be a weighing matrix* $W(p+1, p)$*. Then there is a quasi-residual*

$$BIBD((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})).$$

Proof: Let D_2 be the *mate* residual design to D_1, then $D_1 + D_2 = J_{(p+1)\times 2p}$. Split the weighing matrix W into positive and negative part $W = W^+ - W^-$. Then

$$\mathcal{D} = W^+ \otimes D_1 + W^- \otimes D_2$$

Quasi-residual designs

Lemma

Let $(p+1, 2p, p, \frac{p+1}{2}, \frac{p-1}{2})$ be parameters of a quasi-residual BIBD, say D_1. Let W be a weighing matrix $W(p+1, p)$. Then there is a quasi-residual

$$BIBD((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})).$$

Proof: Let D_2 be the *mate* residual design to D_1, then $D_1 + D_2 = J_{(p+1)\times 2p}$. Split the weighing matrix W into positive and negative part $W = W^+ - W^-$. Then

$$\mathcal{D} = W^+ \otimes D_1 + W^- \otimes D_2$$

is a quasi-residual

$$\mathrm{BIBD}((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})).$$

An example

An example

Example

A $BIBD(16, 30, 15, 8, 7)$ is obtained from the residual design of a $SBIBD(31, 15, 7)$. There is also a weighing matrix $W(16, 15)$. Therefore, there is a quasi-residual $BIBD(256, 480, 225, 120, 105)$.

An example

Example

A $BIBD(16, 30, 15, 8, 7)$ is obtained from the residual design of a $SBIBD(31, 15, 7)$. There is also a weighing matrix $W(16, 15)$. Therefore, there is a quasi-residual $BIBD(256, 480, 225, 120, 105)$.

Theorem

For prime powers p the quasi-residual design $\mathcal{R} = BIBD\left((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})\right)$, *obtained from the residual design of* $P = SBIBD(2p+1, p, \frac{p-1}{2})$ *is embeddable*

An example

Example

A $BIBD(16, 30, 15, 8, 7)$ is obtained from the residual design of a $SBIBD(31, 15, 7)$. There is also a weighing matrix $W(16, 15)$. Therefore, there is a quasi-residual $BIBD(256, 480, 225, 120, 105)$.

Theorem

For prime powers p the quasi-residual design $\mathcal{R} = BIBD\left((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})\right)$, *obtained from the residual design of* $P = SBIBD(2p+1, p, \frac{p-1}{2})$ *is embeddable and a symmetric design* $SBIBD(p^2 + (p+1)^2, p^2, p(p-1)/2)$ *is obtained.*

An example

Example

A $BIBD(16, 30, 15, 8, 7)$ is obtained from the residual design of a $SBIBD(31, 15, 7)$. There is also a weighing matrix $W(16, 15)$. Therefore, there is a quasi-residual $BIBD(256, 480, 225, 120, 105)$.

Theorem

For prime powers p the quasi-residual design $\mathcal{R} = BIBD\left((p+1)^2, 2p(p+1), p^2, p(\frac{p+1}{2}), p(\frac{p-1}{2})\right)$, *obtained from the residual design of* $P = SBIBD(2p+1, p, \frac{p-1}{2})$ *is embeddable and a symmetric design* $SBIBD(p^2 + (p+1)^2, p^2, p(p-1)/2)$ *is obtained.*

A quick summary and more designs in general

A quick summary and more designs in general

Let p be an odd prime power. Consider the seed $P = SBIBD(2p+1, p, \frac{p-1}{2})$. Write

$$P = \left[\begin{array}{c|c} \mathrm{j}_p & \mathcal{D} \\ \hline 0_{p+1} & \mathcal{R} \end{array}\right]$$

A quick summary and more designs in general

Let p be an odd prime power. Consider the seed $P = SBIBD(2p+1, p, \frac{p-1}{2})$. Write

$$P = \left[\begin{array}{c|c} \mathrm{j}_p & \mathcal{D} \\ \hline 0_{p+1} & \mathcal{R} \end{array}\right]$$

By applying a

$$BW\left(\frac{p^{m+1}-1}{p-1}, p^m, p^{m-1}(p-1)\right)$$

there is a

$$SBIBD\left(2p(\frac{p^{m+1}-1}{p-1})+1, p^{m+1}, \frac{p^m(p-1)}{2}\right).$$

A quick summary and more designs in general

Let p be an odd prime power. Consider the seed $P = SBIBD(2p+1, p, \frac{p-1}{2})$. Write

$$P = \left[\begin{array}{c|c} \mathrm{j}_p & \mathcal{D} \\ \hline 0_{p+1} & \mathcal{R} \end{array}\right]$$

By applying a

$$BW\left(\frac{p^{m+1}-1}{p-1}, p^m, p^{m-1}(p-1)\right)$$

there is a

$$SBIBD\left(2p(\frac{p^{m+1}-1}{p-1})+1, p^{m+1}, \frac{p^m(p-1)}{2}\right).$$

The signing of $SBIBD(19, 9, 4)$

The signing of $SBIBD(19, 9, 4)$

Applying the method to the prime power 9

The signing of $SBIBD(19, 9, 4)$

Applying the method to the prime power 9 it follows that there is a symmetric design

$$SBIBD\left(1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m\right).$$

The signing of $SBIBD(19, 9, 4)$

Applying the method to the prime power 9 it follows that there is a symmetric design

$$SBIBD\left(1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m\right).$$

The *signing* of the design will be shown in the next few slides.

The Residual part: $BIBD(10, 18, 9, 5, 4)$

The Residual part: $BIBD(10, 18, 9, 5, 4)$

$$P = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

The Residual part: $BIBD(10, 18, 9, 5, 4)$

$$P = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

- $PP^t = 9I_{10}$.

The Residual part: $BIBD(10,18,9,5,4)$

$$P = \begin{bmatrix}
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1\\
-&0&0&1&0&1&1&1&0&0&0&0&1&0&-&1&-&0\\
0&-&0&1&1&0&0&1&1&0&0&0&-&1&0&0&1&-\\
0&0&-&0&1&1&1&0&1&0&0&0&0&-&1&-&0&1\\
1&1&0&-&0&0&1&0&1&1&-&0&0&0&0&1&0&-\\
0&1&1&0&-&0&1&1&0&0&1&-&0&0&0&-&1&0\\
1&0&1&0&0&-&0&1&1&-&0&1&0&0&0&0&-&1\\
1&0&1&1&1&0&-&0&0&1&0&-&1&-&0&0&0&0\\
1&1&0&0&1&1&0&-&0&-&1&0&0&1&-&0&0&0\\
0&1&1&1&0&1&0&0&-&0&-&1&-&0&1&0&0&0
\end{bmatrix}$$

- $PP^t = 9I_{10}$.
- $|P|$ is the incidence matrix of a $BIBD(10,18,9,5,4)$.

The Residual part: $BIBD(10, 18, 9, 5, 4)$

$$P = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
- & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 \\
0 & - & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - \\
0 & 0 & - & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 \\
1 & 1 & 0 & - & 0 & 0 & 1 & 0 & 1 & 1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - \\
0 & 1 & 1 & 0 & - & 0 & 1 & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 \\
1 & 0 & 1 & 0 & 0 & - & 0 & 1 & 1 & - & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 \\
1 & 0 & 1 & 1 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & - & 0 & - & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 1 & - & 0 & 1 & 0 & 0 & 0
\end{bmatrix}$$

- $PP^t = 9I_{10}$.
- $|P|$ is the incidence matrix of a $BIBD(10, 18, 9, 5, 4)$.

A complementary mate for P

A complementary mate for P

$$Q = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&-&-&0&1&0&0&0&1&-&1&1&0&1&0&0&0&1\\
-&0&-&0&0&1&1&0&0&1&-&1&0&0&1&1&0&0\\
-&-&0&1&0&0&0&1&0&1&1&-&1&0&0&0&1&0\\
0&0&1&0&-&-&0&1&0&0&0&1&-&1&1&0&1&0\\
1&0&0&-&0&-&0&0&1&1&0&0&1&-&1&0&0&1\\
0&1&0&-&-&0&1&0&0&0&1&0&1&1&-&1&0&0\\
0&1&0&0&0&1&0&-&-&0&1&0&0&0&1&-&1&1\\
0&0&1&1&0&0&-&0&-&0&0&1&1&0&0&1&-&1\\
1&0&0&0&1&0&-&-&0&1&0&0&0&1&0&1&1&-
\end{bmatrix}$$

A complementary mate for P

$$Q = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
- & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
- & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{bmatrix}$$

- $QQ^t = 9I_{10}$.

A complementary mate for P

$$
Q = \begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\
- & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
- & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\
1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\
0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\
1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & -
\end{bmatrix}
$$

- $QQ^t = 9I_{10}$.
- $|Q|$ is the incidence matrix of a $BIBD(10, 18, 9, 5, 4)$.

A complementary mate for P

$$Q = \begin{bmatrix}
1&1&1&1&1&1&1&1&1&0&0&0&0&0&0&0&0&0\\
0&-&-&0&1&0&0&0&1&-&1&1&0&1&0&0&0&1\\
-&0&-&0&0&1&1&0&0&1&-&1&0&0&1&1&0&0\\
-&-&0&1&0&0&0&1&0&1&1&-&1&0&0&0&1&0\\
0&0&1&0&-&-&0&1&0&0&0&1&-&1&1&0&1&0\\
1&0&0&-&0&-&0&0&1&1&0&0&1&-&1&0&0&1\\
0&1&0&-&-&0&1&0&0&0&1&0&1&1&-&1&0&0\\
0&1&0&0&0&1&0&-&-&0&1&0&0&0&1&-&1&1\\
0&0&1&1&0&0&-&0&-&0&0&1&1&0&0&1&-&1\\
1&0&0&0&1&0&-&-&0&1&0&0&0&1&0&1&1&-
\end{bmatrix}$$

- $QQ^t = 9I_{10}$.
- $|Q|$ is the incidence matrix of a $BIBD(10, 18, 9, 5, 4)$.

A Derived part for both P and Q

A Derived part for both P and Q

There is a signed $D = BIBD(9, 18, 8, 4, 3)$ that when added to either P or Q it turns them into a $BW(19, 9, 4)$.

A Derived part for both P and Q

There is a signed $D = BIBD(9, 18, 8, 4, 3)$ that when added to either P or Q it turns them into a $BW(19, 9, 4)$.

$$D = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
- & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
- & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{bmatrix}$$

A Derived part for both P and Q

There is a signed $D = BIBD(9, 18, 8, 4, 3)$ that when added to either P or Q it turns them into a $BW(19, 9, 4)$.

$$D = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & - & 1 & - & 0 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & - & 1 & 0 & 0 & 1 & - & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & - & 1 & - & 0 & 1 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & - & 0 & 0 & 0 & 0 & 1 & 0 & - & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 \\
0 & 1 & - & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 \\
- & 0 & 1 & 0 & 0 & 0 & 0 & - & 1 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 \\
1 & 0 & - & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - \\
- & 1 & 0 & 0 & 1 & - & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - \\
0 & - & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0
\end{bmatrix}$$

Quaternary balanced generalized weighing matrices

Quaternary balanced generalized weighing matrices

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & i & - & j & 1 & i & - & j & 1 \\ 1 & i & 0 & j & - & j & i & 1 & 1 & - \\ 1 & - & j & 0 & 1 & j & 1 & - & i & i \\ 1 & j & - & 1 & 0 & i & 1 & i & j & - \\ 1 & 1 & j & j & i & 0 & - & i & - & 1 \\ 1 & i & i & 1 & 1 & - & 0 & j & - & j \\ 1 & - & 1 & - & i & i & j & 0 & 1 & j \\ 1 & j & 1 & i & j & - & - & 1 & 0 & i \\ 1 & 1 & - & i & - & 1 & j & j & i & 0 \end{bmatrix} = W_0 + iW_1 + (-1)W_2 + (-i)W_3$$

Quaternary balanced generalized weighing matrices

$$
\begin{bmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & i & - & j & 1 & i & - & j & 1 \\
1 & i & 0 & j & - & j & i & 1 & 1 & - \\
1 & - & j & 0 & 1 & j & 1 & - & i & i \\
1 & j & - & 1 & 0 & i & 1 & i & j & - \\
1 & 1 & j & j & i & 0 & - & i & - & 1 \\
1 & i & i & 1 & 1 & - & 0 & j & - & j \\
1 & - & 1 & - & i & i & j & 0 & 1 & j \\
1 & j & 1 & i & j & - & - & 1 & 0 & i \\
1 & 1 & - & i & - & 1 & j & j & i & 0
\end{bmatrix}
= W_0 + iW_1 + (-1)W_2 + (-i)W_3
$$

Quaternary balanced generalized weighing matrices

Quaternary balanced generalized weighing matrices

Let W be a BGW$(\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1})$ over $\mathbb{Z}_4$ generated by the complex unit i. There are 4 disjoint (0,1)-matrices $\{W_i\}_{i=0}^{3}$ satisfying

$$W = \sum_{j=0}^{3} (i)^j W_j.$$

Quaternary balanced generalized weighing matrices

Let W be a BGW$(\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1})$ over $\mathbb{Z}_4$ generated by the complex unit i. There are 4 disjoint (0,1)-matrices $\{W_i\}_{i=0}^{3}$ satisfying

$$W = \sum_{j=0}^{3} (i)^j W_j.$$

Define the $10\frac{9^{m+1}-1}{8} \times 18\frac{9^{m+1}-1}{8}$ matrix $\mathcal{R}$ as

$$\mathcal{R} = W_0 \otimes P + W_1 \otimes (-Q) + W_2 \otimes (-P) + W_3 \otimes Q.$$

Quaternary balanced generalized weighing matrices

Let W be a BGW$(\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1})$ over $\mathbb{Z}_4$ generated by the complex unit i. There are 4 disjoint (0,1)-matrices $\{W_i\}_{i=0}^{3}$ satisfying

$$W = \sum_{j=0}^{3} (i)^j W_j.$$

Define the $10\frac{9^{m+1}-1}{8} \times 18\frac{9^{m+1}-1}{8}$ matrix $\mathcal{R}$ as

$$\mathcal{R} = W_0 \otimes P + W_1 \otimes (-Q) + W_2 \otimes (-P) + W_3 \otimes Q.$$

Then,

- $\mathcal{R}\mathcal{R}^t = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.

Quaternary balanced generalized weighing matrices

Let W be a BGW$(\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1})$ over $\mathbb{Z}_4$ generated by the complex unit i. There are 4 disjoint (0,1)-matrices $\{W_i\}_{i=0}^{3}$ satisfying

$$W = \sum_{j=0}^{3} (i)^j W_j.$$

Define the $10\frac{9^{m+1}-1}{8} \times 18\frac{9^{m+1}-1}{8}$ matrix $\mathcal{R}$ as

$$\mathcal{R} = W_0 \otimes P + W_1 \otimes (-Q) + W_2 \otimes (-P) + W_3 \otimes Q.$$

Then,

- $\mathcal{R}\mathcal{R}^t = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.

Quaternary balanced generalized weighing matrices

Let W be a BGW$(\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1})$ over $\mathbb{Z}_4$ generated by the complex unit i. There are 4 disjoint (0,1)-matrices $\{W_i\}_{i=0}^{3}$ satisfying

$$W = \sum_{j=0}^{3} (i)^j W_j.$$

Define the $10\frac{9^{m+1}-1}{8} \times 18\frac{9^{m+1}-1}{8}$ matrix $\mathcal{R}$ as

$$\mathcal{R} = W_0 \otimes P + W_1 \otimes (-Q) + W_2 \otimes (-P) + W_3 \otimes Q.$$

Then,

- $\mathcal{R}\mathcal{R}^t = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.

The outcome

Lemma

1. $RR^\top = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.

The outcome

Lemma

1. $RR^\top = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.
2. $DD^\top = 9^{m+1} I_{9^{m+1}} - J_{9^{m+1}}$.

The outcome

Lemma

1. $RR^\top = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.
2. $DD^\top = 9^{m+1} I_{9^{m+1}} - J_{9^{m+1}}$.
3. $RD^\top = DR^\top = 0$.

The outcome

Lemma

1. $RR^\top = 9^{m+1} I_{\frac{10(9^{m+1}-1)}{8}}$.
2. $DD^\top = 9^{m+1} I_{9^{m+1}} - J_{9^{m+1}}$.
3. $RD^\top = DR^\top = 0$.

The design is signed

The design is signed

The matrix

$$\left[\begin{array}{c|c} \mathrm{j}_{p^{m+1}} & \mathcal{D} \\ \hline 0_{18\frac{9^{m+1}-1}{8}} & \mathcal{R} \end{array}\right]$$

The design is signed

The matrix

$$\left[\begin{array}{c|c} \mathrm{j}_{p^{m+1}} & \mathcal{D} \\ \hline 0_{18\frac{9^{m+1}-1}{8}} & \mathcal{R} \end{array}\right]$$

is the desired balanced weighing matrix and:

NSERC CRSNG

The design is signed

The matrix

$$\left[\begin{array}{c|c} \mathrm{j}_{p^{m+1}} & \mathcal{D} \\ \hline 0_{18\frac{9^{m+1}-1}{8}} & \mathcal{R} \end{array}\right]$$

is the desired balanced weighing matrix and:

Theorem

There is a balanced weighing matrix with parameters

$$BW\left(1 + 18 \cdot \frac{9^{m+1}-1}{8}, 9^{m+1}, 4 \cdot 9^m\right)$$

for each nonzero integer m.

The design is signed

The matrix

$$\left[\begin{array}{c|c} \mathrm{j}_{p^{m+1}} & \mathcal{D} \\ \hline 0_{18\frac{9^{m+1}-1}{8}} & \mathcal{R} \end{array}\right]$$

is the desired balanced weighing matrix and:

Theorem

There is a balanced weighing matrix with parameters

$$BW\left(1+18\cdot\frac{9^{m+1}-1}{8}, 9^{m+1}, 4\cdot 9^{m}\right)$$

for each nonzero integer m.

Two open questions

1. A necessary and sufficient condition for the existence of a weighing matrix

$$W(n,k) = \left[\begin{array}{c|cccccc}
0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\
\hline
1 & & & & & & \\
 & 0 & & & & & * \\
\vdots & & & & & & \\
 & & & \ddots & & & \\
1 & & & & & & \\
0 & & & & & & \\
 & & & & \ddots & & \\
\vdots & & & & & & \\
 & * & & & & & 0 \\
0 & & & & & &
\end{array}\right]$$

Two open questions

1. A necessary and sufficient condition for the existence of a weighing matrix

$$W(n,k) = \left[\begin{array}{c|cccccc} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & & & \\ & 0 & & & & & * \\ \vdots & & & & & & \\ & & & \ddots & & & \\ 1 & & & & & & \\ 0 & & & & \ddots & & \\ \vdots & & & & & & \\ & * & & & & & 0 \\ 0 & & & & & & \end{array}\right]$$

2. Is there a balanced $BW(2q^2+1, q^2, \frac{q^2-1}{2})$ for every prime power q?

Two open questions

1. A necessary and sufficient condition for the existence of a weighing matrix

$$W(n,k) = \left[\begin{array}{c|cccccc} 0 & 1 & \cdots & 1 & 0 & \cdots & 0 \\ \hline 1 & & & & & & \\ \vdots & & 0 & & & & * \\ 1 & & & \ddots & & & \\ 0 & & & & \ddots & & \\ \vdots & & & & & & \\ 0 & & * & & & & 0 \end{array}\right]$$

2. Is there a balanced $BW(2q^2+1, q^2, \frac{q^2-1}{2})$ for every prime power q?

The first open case is the existence of a $BW(51, 25, 12)$.