

# Constructing some combinatorial matrices by using orthogonal arrays

Combinatorial Designs and Codes

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Joint work with Thomas Pender and Sho Suda

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## A weighing matrix $W(19, 9)$

$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & -0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -0 & & & \\ 0 & 0 & -0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & - & & \\ 0 & 0 & 0 & -0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -0 & 1 & & & \\ 0 & 1 & 1 & 0 & -0 & 0 & 1 & 0 & 1 & 1 & -0 & 0 & 0 & 0 & 1 & 0 & - & & \\ 0 & 0 & 1 & 1 & 0 & -0 & 1 & 1 & 0 & 0 & 1 & -0 & 0 & 0 & -1 & 0 & & & \\ 0 & 1 & 0 & 1 & 0 & 0 & -0 & 1 & 1 & -0 & 1 & 0 & 0 & 0 & 0 & -1 & & & \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & -0 & 0 & 1 & 0 & -1 & -0 & 0 & 0 & 0 & & & \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & -0 & -1 & 0 & 0 & 1 & -0 & 0 & 0 & & & \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -0 & -1 & -0 & 1 & 0 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & 1 & 0 & -1 & -0 & 0 & - & 0 & 1 & 0 & 0 & 0 & 1 & & & \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & - & 0 & -0 & 0 & 1 & 1 & 0 & 0 & & & \\ 1 & 0 & 0 & 0 & 0 & -1 & -0 & 1 & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & & & \\ 1 & 1 & -0 & 0 & 0 & 0 & 1 & 0 & -0 & 0 & 1 & 0 & - & 0 & 1 & 0 & & & \\ 1 & 0 & 1 & -0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 & -0 & -0 & 0 & 0 & 1 & & & \\ 1 & -0 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & - & 0 & 1 & 0 & 0 & & & \\ 1 & 1 & 0 & -1 & -0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & & & \\ 1 & -1 & 0 & 0 & 1 & -0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & -0 & - & & & \\ 1 & 0 & -1 & -0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & 0 & & & \end{bmatrix}$$



# Weighing matrices



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in such a way that every two different rows have exactly

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Starting with a  $W(8, 5)$  (a seed weighing matrix),  $n = 8$ ,  $p = 5$  and  $m = 1$ , a  $W(43, 25)$  is constructed.

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Here is a weighing matrix  $W(8, 5)$ :

$$H' = \left[ \begin{array}{c|cccccc} 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & - & - & 0 & 1 & 0 & 0 \\ 1 & - & 1 & - & 0 & 0 & 1 & 0 \\ 1 & - & - & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & - & - & - & - \\ \hline 0 & 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 0 & 1 & - & 1 & 1 & - \end{array} \right].$$

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Then

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & - & - & 0 & 1 & 0 & 0 \\ - & 1 & - & 0 & 0 & 1 & 0 \\ - & - & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & - & - & - & - \end{bmatrix},$$

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$$R = \begin{bmatrix} 1 & 0 & 0 & - & - & 1 & 1 \\ 0 & 1 & 0 & - & 1 & - & 1 \\ 0 & 0 & 1 & - & 1 & 1 & - \end{bmatrix},$$

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# The Orthogonal design O

1	1	1	1	1	1
1	2	2	2	2	2
1	3	3	3	3	3
1	4	4	4	4	4
1	5	5	5	5	5
2	1	2	3	4	5
2	2	3	4	5	1
2	3	4	5	1	2
2	4	5	1	2	3
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Replacing the five numbers in the array  $O$  with the five rows of  $R$  (in any order) a matrix  $\mathcal{D}$  is obtained for which  $\mathcal{D}\mathcal{D}^t = 25 - J$ .





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If  $W$  is any  $W(6, 5)$ , then

$$\mathcal{R} = W \otimes R = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & - & - & 1 \\ 1 & 1 & 0 & 1 & - & - \\ 1 & - & 1 & 0 & 1 & - \\ 1 & - & - & 1 & 0 & 1 \\ 1 & 1 & - & - & 1 & 0 \end{bmatrix} \otimes R = \begin{bmatrix} 0 & R & R & R & R & R \\ R & 0 & R & \bar{R} & \bar{R} & R \\ R & R & 0 & R & \bar{R} & \bar{R} \\ R & \bar{R} & R & 0 & R & \bar{R} \\ R & \bar{R} & \bar{R} & R & 0 & R \\ R & R & \bar{R} & \bar{R} & R & 0 \end{bmatrix}$$

and  $\mathcal{R}\mathcal{R}^t = 25I$ . Moreover,  $\mathcal{D}\mathcal{R}^t = 0$ .

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forms a  $W(43, 25)$ .

## Constructing $W(43, 25)$

Replacing the five numbers in the array  $O$  with the five rows of  $R$  (in any order) a matrix  $\mathcal{D}$  is obtained for which  $\mathcal{D}\mathcal{D}^t = 25I - J$ .  
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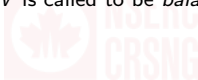


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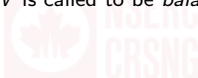


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1	0	0	0	-	1	0	0	1	-	-	0	-	0	0	1	1	0	0	0
1	0	0	0	0	-	1	-	0	1	-	-	0	1	0	0	0	0	1	0
1	1	-	0	0	0	0	1	0	-	0	0	1	0	-	-	0	1	0	0
1	0	1	-	0	0	0	-	1	0	1	0	0	-	0	-	0	0	1	0
1	-	0	1	0	0	0	0	-	1	0	1	0	-	-	0	1	0	0	0
1	1	0	-	1	-	0	0	0	0	0	1	0	0	0	1	0	-	-	-
1	-	1	0	0	1	-	0	0	0	0	0	1	1	0	0	-	0	-	-
1	0	-	1	-	0	1	0	0	0	1	0	0	0	1	0	-	-	0	0
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0	0	-	-	0	1	0	0	0	1	-	1	1	0	1	0	0	0	1	0
0	-	0	-	0	0	1	1	0	0	1	-	1	0	0	1	1	0	0	0
0	-	-	0	1	0	0	0	1	0	1	1	-	1	0	0	0	1	0	0
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0	1	0	0	-	0	-	0	0	1	1	0	0	1	-	1	0	0	1	0
0	0	1	0	-	-	0	1	0	0	0	1	0	1	1	-	1	0	0	0
0	0	1	0	0	0	1	0	-	-	0	1	0	0	0	1	-	1	1	1
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1	1	-	0	0	0	0	1	0	-	0	0	1	0	-	-	0	1	0	0
1	0	1	-	0	0	0	-	1	0	1	0	0	-	0	-	0	0	1	0
1	-	0	1	0	0	0	0	-	1	0	1	0	-	-	0	1	0	0	0
1	1	0	-	1	-	0	0	0	0	0	1	0	0	0	1	0	-	-	-
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0	0	1	0	-	-	0	1	0	0	0	1	0	1	1	-	1	0	0	0
0	0	1	0	0	0	1	0	-	-	0	1	0	0	0	1	-	1	1	1
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In order to construct the infinite class we need to show

- a method to construct an infinite number of symmetric designs from a single seed symmetric design.
- a method to *sign* the symmetric design.

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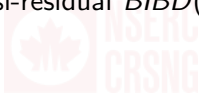
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# The signing of $SBIBD(19, 9, 4)$



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Applying the method to the prime power 9



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Applying the method to the prime power 9 it follows that there is a symmetric design

$$SBIBD \left( 1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m \right).$$

## The signing of $SBIBD(19, 9, 4)$

Applying the method to the prime power 9 it follows that there is a symmetric design

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The *signing* of the design will be shown in the next few slides.



The Residual part:  $BIBD(10, 18, 9, 5, 4)$



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$$P = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ -0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -0 & & & \\ 0 & -0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & - & & \\ 0 & 0 & -0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & -0 & 1 & & & & \\ 1 & 1 & 0 & -0 & 0 & 1 & 0 & 1 & 1 & -0 & 0 & 0 & 0 & 1 & 0 & - & & & \\ 0 & 1 & 1 & 0 & -0 & 1 & 1 & 0 & 0 & 1 & -0 & 0 & 0 & -1 & 0 & & & & \\ 1 & 0 & 1 & 0 & 0 & -0 & 1 & 1 & -0 & 1 & 0 & 0 & 0 & 0 & -1 & & & & \\ 1 & 0 & 1 & 1 & 1 & 0 & -0 & 0 & 1 & 0 & -1 & -0 & 0 & 0 & 0 & & & & \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & -0 & -1 & 0 & 0 & 1 & -0 & 0 & 0 & & & & \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & -0 & -1 & -0 & 1 & 0 & 0 & 0 & & & & \end{bmatrix}$$

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- $PP^t = 9I_{10}$ .

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A complementary mate for  $P$



## A complementary mate for $P$

$$Q = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & - & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & - & 0 & - & 0 & 0 & 1 & 1 & 0 & 0 & 1 & - & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & - & - & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & - \end{bmatrix}$$

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A Derived part for both  $P$  and  $Q$



## A Derived part for both $P$ and $Q$

**There is a signed  $D = BIBD(9, 18, 8, 4, 3)$  that when added to either  $P$  or  $Q$  it turns them into a  $BW(19, 9, 4)$ .**



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# Quaternary balanced generalized weighing matrices



## Quaternary balanced generalized weighing matrices

$$\begin{bmatrix}
 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
 1 & 0 & i & -j & 1 & i & -j & 1 & & \\
 1 & i & 0 & j & -j & i & 1 & 1 & - & \\
 1 & -j & 0 & 1 & j & 1 & -i & i & & \\
 1 & j & -1 & 0 & i & 1 & i & j & - & \\
 1 & 1 & j & j & i & 0 & -i & -1 & & \\
 1 & i & i & 1 & 1 & -0 & j & -j & & \\
 1 & -1 & -i & i & j & 0 & 1 & j & & \\
 1 & j & 1 & i & j & - & -1 & 0 & i & \\
 1 & 1 & -i & -1 & j & j & i & 0 & & 
 \end{bmatrix} = W_0 + iW_1 + (-1)W_2 + (-i)W_3$$



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 1 & j & -1 & 0 & i & 1 & i & j & - & \\
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# Quaternary balanced generalized weighing matrices



## Quaternary balanced generalized weighing matrices

Let  $W$  be a BGW( $\frac{9^{m+1}-1}{8}, 9^m, 9^m - 9^{m-1}$ ) over  $\mathbb{Z}_4$  generated by the complex unit  $i$ . There are 4 disjoint  $(0,1)$ -matrices  $\{W_i\}_{i=0}^3$  satisfying

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$$\mathcal{R} = W_0 \otimes P + W_1 \otimes (-Q) + W_2 \otimes (-P) + W_3 \otimes Q.$$

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# The outcome

Lemma

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The design is signed



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The matrix

$$\left[ \begin{array}{c|c} j_{p^{m+1}} & \mathcal{D} \\ \hline 0_{18 \frac{9^{m+1}-1}{8}} & \mathcal{R} \end{array} \right]$$

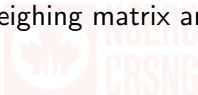


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### Theorem

*There is a balanced weighing matrix with parameters*

$$BW \left( 1 + 18 \cdot \frac{9^{m+1} - 1}{8}, 9^{m+1}, 4 \cdot 9^m \right)$$

*for each nonzero integer  $m$ .*

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## Two open questions

1. A necessary and sufficient condition for the existence of a weighing matrix

$$W(n, k) = \left[ \begin{array}{c|cccccc} 0 & 1 & \dots & 1 & 0 & \dots & 0 \\ \hline 1 & & & & & & \\ \vdots & 0 & & & & & * \\ 1 & & & \ddots & & & \\ 0 & & & & & \ddots & \\ \vdots & & & & & & \\ 0 & * & & & & & 0 \end{array} \right]$$

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The first open case is the existence of a  $BW(51, 25, 12)$ .