On 4-designs with three intersection numbers*

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(joint work with Renata Vlahović Kruc)

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Combinatorial Designs and Codes

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The strength and the degree of a design

Let V be a set of v points. A design $\mathcal{B}$ is a family of k-subsets of V called blocks. The strength of $\mathcal{B}$ is the maximal t such that $\mathcal{B}$ is a t-(v, k, λ) design for some λ. Number of blocks: $b = |\mathcal{B}|$, number of blocks through a point: r.

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Designs with $t = 2d$ are called tight and have exactly $b = \binom{v}{d}$ blocks.

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$d = 2,\ t = 4 \quad \rightsquigarrow \quad$ Derived Witt design $4\text{-}(23, 7, 1)$ and its complement.

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$d = 2,\ t = 3$

$x = 0 \ \rightsquigarrow$ Extensions of symmetric designs, classified in:

P. J. Cameron, *Extending symmetric designs*, J. Combinatorial Theory Ser. A **14** (1973), 215–220.

Tight and not so tight designs

- 3-$(4(\lambda+1), 2(\lambda+1), \lambda)$ (Hadamard 3-designs),
- 3-$((\lambda+1)(\lambda^2+5\lambda+5), (\lambda+1)(\lambda+2), \lambda)$,
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A. E. Brouwer, H. Van Maldeghem, *Strongly regular graphs*, 2021.
https://homepages.cwi.nl/~aeb/math/srg/rk3/srgw.pdf

Quasi-symmetric 2-designs

v	k	λ	y	x	ex
19	7	7	1	3	=
19	9	16	3	5	=
20	8	14	2	4	=
20	10	18	4	6	=
21	6	4	0	2	!
21	7	12	1	3	!
21	8	14	2	4	–
21	9	12	3	5	–
22	6	5	0	2	!
22	7	16	1	3	!
22	8	12	2	4	–
23	7	21	1	3	!
24	8	7	2	4	–
28	7	16	1	3	–
28	12	11	4	6	+
29	7	12	1	3	–
31	7	7	1	3	5
33	9	6	1	3	–
33	15	35	6	9	?
35	7	3	1	3	–
35	14	13	5	8	?
36	16	12	6	8	+
37	9	8	1	3	–
39	12	22	3	6	?
41	9	9	1	3	?
41	17	34	5	8	–

v	k	λ	y	x	ex
41	20	57	8	11	?
42	18	51	6	9	?
42	21	60	9	12	?
43	16	40	4	7	?
43	18	51	6	9	–
45	9	8	1	3	!
45	15	42	3	6	?
45	18	34	6	9	?
45	21	70	9	13	?
46	16	8	4	6	?
46	16	72	4	7	?
49	9	6	1	3	+
49	13	13	1	4	?
49	16	45	4	7	?
51	15	7	3	5	–
51	21	14	6	9	–
52	16	20	4	7	–
55	15	7	3	5	?
55	15	63	3	6	?
55	16	40	4	8	?
56	12	9	0	3	–
56	15	42	3	6	–
56	16	6	4	6	+
56	16	18	4	8	+
56	20	19	5	8	?
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28	7	16	1	3	–
28	12	11	4	6	+
29	7	12	1	3	–
31	7	7	1	3	5
33	9	6	1	3	–
33	15	35	6	9	?
35	7	3	1	3	–
35	14	13	5	8	?
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The eigenvalues of the graph can be computed from the design parameters 2-(v, k, λ) and the block intersection numbers x, y:

$$\theta_0 = \frac{k(r-1) - x(b-1)}{y-x}, \quad \theta_1 = \frac{r - \lambda - k + x}{y-x}, \quad \theta_2 = \frac{x-k}{y-x}.$$

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These in turn determine parameters of the block graph $SRG(b, \theta_0, \overline{\lambda}, \overline{\mu})$:

$$\overline{\lambda} = \theta_0 + \theta_1 + \theta_2 + \theta_1\theta_2, \qquad \overline{\mu} = \theta_0 + \theta_1\theta_2.$$

Designs of degree $d = 3$

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$d = 3$, $t = 5$ $\rightsquigarrow$ The only examples are hypothesized to be the Witt design 5-$(24, 8, 1)$ and its complement.

Y. J. Ionin, M. S. Shrikhande, 5-*designs with three intersection numbers*, J. Combin. Theory Ser. A **69** (1995), no. 1, 36–50.

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$d = 3,\ t = 4$ $\rightsquigarrow$ The Cameron-Deslarte theorem still applies!

The three block graphs form a symmetric 3-class association scheme.

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The three block graphs form a symmetric 3-class association scheme.

Eigenvalues of the scheme can be computed from the design parameters 4-(v, k, λ) and $D = \{x, y, z\}$. They determine intersection numbers of the scheme and the Krein parameters, giving many nontrivial conditions (integrality, non-negativity, absolute bound).

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$
1	11	5	1	1	2	3	
2	23	8	4	0	2	4	
3	23	11	48	3	5	7	
4	24	8	5	0	2	4	
5	47	11	8	1	3	5	
6	71	35	264	14	17	20	
7	199	99	2328	44	49	54	
8	391	195	9264	90	97	104	
9	647	323	25680	152	161	170	
10	659	329	390874	153	164	175	
11	967	483	57720	230	241	252	

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No.	v	k	λ	x	y	z	$\exists$
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2	23	8	4	0	2	4	
3	23	11	48	3	5	7	
4	24	8	5	0	2	4	
5	47	11	8	1	3	5	
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2	23	8	4	0	2	4		
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3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	$\lvert\mathbb{Z}_{47} \rtimes \mathbb{Z}_{23}\rvert = 1081$
6	71	35	264	14	17	20		
7	199	99	2328	44	49	54		
8	391	195	9264	90	97	104		
9	647	323	25680	152	161	170		
10	659	329	390874	153	164	175		
11	967	483	57720	230	241	252		

Feasible parameters 4-(v,k,λ) for $d=3$ and $v \le 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-(12, 6, 1)
2	23	8	4	0	2	4	✓	res 5-(24, 8, 1)
3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	der 5-(48, 12, 8), $\vert PGL(2,47)\vert = 103776$
6	71	35	264	14	17	20		
7	199	99	2328	44	49	54		
8	391	195	9264	90	97	104		
9	647	323	25680	152	161	170		
10	659	329	390874	153	164	175		
11	967	483	57720	230	241	252		

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-$(12, 6, 1)$
2	23	8	4	0	2	4	✓	res 5-$(24, 8, 1)$
3	23	11	48	3	5	7	✓	der 5-$(24, 12, 48)$
4	24	8	5	0	2	4	✓	5-$(24, 8, 1)$
5	47	11	8	1	3	5	✓	der 5-$(48, 12, 8)$,
6	71	35	264	14	17	20		$\lvert PGL(2, 47)\rvert = 103776$
7	199	99	2328	44	49	54		
8	391	195	9264	90	97	104		
9	647	323	25680	152	161	170		
10	659	329	390874	153	164	175		
11	967	483	57720	230	241	252		

V. D. Tonchev, *Quasi-symmetric* 2-$(31, 7, 7)$ *designs and a revision of Hamada's conjecture*, J. Combin. Theory Ser. A **42** (1986), no. 1, 104–110.

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-(12, 6, 1)
2	23	8	4	0	2	4	✓	res 5-(24, 8, 1)
3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	der 5-(48, 12, 8)
6	71	35	264	14	17	20		
7	199	99	2328	44	49	54		
8	391	195	9264	90	97	104		
9	647	323	25680	152	161	170		
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11	967	483	57720	230	241	252		

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-(12, 6, 1)
2	23	8	4	0	2	4	✓	res 5-(24, 8, 1)
3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	der 5-(48, 12, 8)
6	71	35	264	14	17	20	?	
7	199	99	2328	44	49	54	?	
8	391	195	9264	90	97	104	?	
9	647	323	25680	152	161	170	?	
10	659	329	390874	153	164	175	?	
11	967	483	57720	230	241	252	?	

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-$(12, 6, 1)$
2	23	8	4	0	2	4	✓	res 5-$(24, 8, 1)$
3	23	11	48	3	5	7	✓	der 5-$(24, 12, 48)$
4	24	8	5	0	2	4	✓	5-$(24, 8, 1)$
5	47	11	8	1	3	5	✓	der 5-$(48, 12, 8)$
6	71	35	264	14	17	20	?	
7	199	99	2328	44	49	54	?	
8	391	195	9264	90	97	104	?	
9	647	323	25680	152	161	170	?	
10	659	329	390874	153	164	175	?	
11	967	483	57720	230	241	252	?	

der 5-$(v+1, k+1, \lambda)$

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-(12, 6, 1)
2	23	8	4	0	2	4	✓	res 5-(24, 8, 1)
3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	der 5-(48, 12, 8)
6	71	35	264	14	17	20	?	der 5-$(v+1, k+1, \lambda)$, $v = 2k+1$
7	199	99	2328	44	49	54	?	
8	391	195	9264	90	97	104	?	
9	647	323	25680	152	161	170	?	
10	659	329	390874	153	164	175	?	
11	967	483	57720	230	241	252	?	

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-$(12, 6, 1)$
2	23	8	4	0	2	4	✓	res 5-$(24, 8, 1)$
3	23	11	48	3	5	7	✓	der 5-$(24, 12, 48)$
4	24	8	5	0	2	4	✓	5-$(24, 8, 1)$
5	47	11	8	1	3	5	✓	der 5-$(48, 12, 8)$
6	71	35	264	14	17	20	?	der 5-$(v+1, k+1, \lambda)$, $v = 2k+1$
7	199	99	2328	44	49	54	?	
8	391	195	9264	90	97	104	?	
9	647	323	25680	152	161	170	?	
10	659	329	390874	153	164	175	?	
11	967	483	57720	230	241	252	?	

G_x is distance-regular of diameter 3 $\rightsquigarrow$ metric (P-polynomial) scheme

Feasible parameters 4-(v, k, λ) for $d = 3$ and $v \leq 1000$

No.	v	k	λ	x	y	z	$\exists$	
1	11	5	1	1	2	3	✓	der 5-(12, 6, 1)
2	23	8	4	0	2	4	✓	res 5-(24, 8, 1)
3	23	11	48	3	5	7	✓	der 5-(24, 12, 48)
4	24	8	5	0	2	4	✓	5-(24, 8, 1)
5	47	11	8	1	3	5	✓	der 5-(48, 12, 8)
6	71	35	264	14	17	20	?	
7	199	99	2328	44	49	54	?	
8	391	195	9264	90	97	104	?	
9	647	323	25680	152	161	170	?	
10	659	329	390874	153	164	175	?	
11	967	483	57720	230	241	252	?	

Rows 6–11: der 5-$(v+1, k+1, \lambda)$, $v = 2k+1$

G_y is strongly regular, $\theta_{1,2} = \pm k$, $\theta_{1,2} \neq \pm k$

The End

Thanks for your attention!