Extending perfect matchings to hamiltonian cycles in line graphs.

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Background

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- **TRUE** for $n = 2, 3, 4$ (Fink, 2007) and for $n = 5$ (Wang and Zhao, 2018).

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- **TRUE** for $n = 2, 3, 4$ (Fink, 2007) and for $n = 5$ (Wang and Zhao, 2018).
- (Fink, 2007) Perfect matchings extend to hamiltonian cycles in Q_n.

Pairing-Hamiltonian-Property (or the PH-property)

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Definition

Let G be a graph having an even number of vertices. Let K_G be the complete graph on $V(G)$. A pairing of G is a perfect matching of K_G.

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Theorem (Fink, 2007)

Q_n has the PH-property.

Results on the PH-property

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Theorem (Thomassen et al., 2015)

Let G be a cubic graph with the PH-property. Then G is either:

- *the complete graph K_4, or*
- *the complete bipartite graph $K_{3,3}$, or*
- *the 3-cube.*

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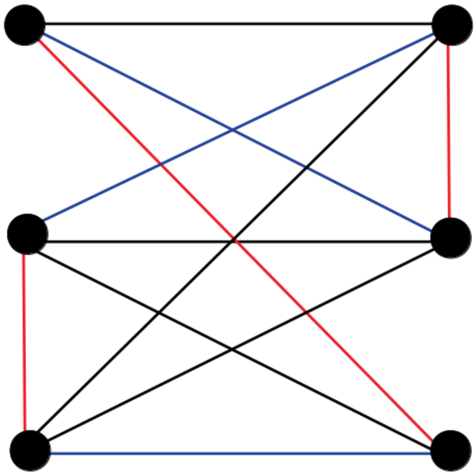

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Since every perfect matching of G is a pairing of G:

- PH-Property $\Rightarrow$ PMH-property

PMH-Property in Literature

In literature, graphs having the PMH-Property are known as F-Hamiltonian graphs. Some known results:

Theorem (Las Vergnas, 1972)

Let G be a bipartite graph, with partite sets V_1 and V_2, such that $|V_1| = |V_2| = m \geq 2$. If for each pair of non-adjacent vertices $u_1 \in V_1$ and $u_2 \in V_2$ we have:

$$deg(u_1) + deg(u_2) \geq m + 1,$$

then G has the PMH-property.

PMH-Property in Literature

Theorem (Haggkvist, 1979)

Let G be a graph, such that $|V(G)|$ is even and at least 4. If for each pair of non-adjacent vertices u and v we have:

$$deg(u) + deg(v) \geq |V(G)| + 1,$$

then G has the PMH-property.

PMH-Property in Literature

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Most recent results we could find (apart from Fink's) are:

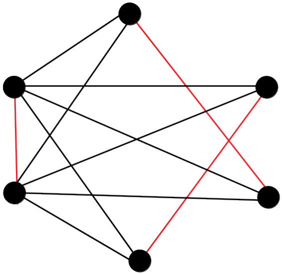
PMH-Property in Literature

Theorem (Z.Yang, 1999)

Let G be a graph on $n \geq 4$ vertices, $n \equiv 0 \mod 2$, $\min_{v \in V(G)}\{deg(v)\} \geq 2$, and

$$|E(G)| \geq \frac{(n-1)(n-2)}{2} + 1.$$

Then, G has the PMH-property if and only if G is not the following graph:



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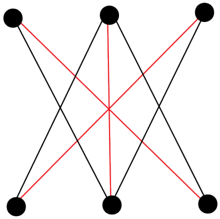

Line graphs of graphs with small maximum degree

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- Which are the graphs whose line graph admits the PMH-property?

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- (Harary, Nash–Williams, 1965): $L(G)$ is hamiltonian if and only if G admits a dominating tour.

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- (Harary, Nash–Williams, 1965): $L(G)$ is hamiltonian if and only if G admits a dominating tour.
- Recall that a dominating tour is a tour in which every edge of G is incident with at least one vertex of the tour.
- This implies that if G is hamiltonian or eulerian, then, $L(G)$ is also hamiltonian.

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Theorem (Abreu, Gauci, DL, Mazzuoccolo, Zerafa, El.J. Combin. 2021)

Let G be a hamiltonian graph such that $\Delta(G) \leq 3$, then $L(G)$ is PMH.

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- In particular Theorem applies for all hamiltonian cubic graphs.
- In the cubic case we can say more:
- (Kotzig, 1964): the existence of a hamiltonian cycle in a cubic graph is both a necessary and sufficient condition for a partition of $L(G)$ into two hamiltonian cycles.

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Corollary (Abreu, Gauci, DL, Mazzuoccolo, Zerafa, El.J.Combin. 2021)

G be a hamiltonian cubic graph and M a perfect matching of $L(G)$. Then, $L(G)$ can be partitioned into two hamiltonian cycles, one of which contains M.

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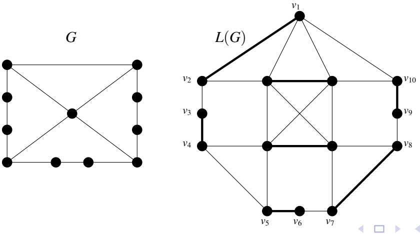

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Proposition (Abreu, Gauci, DL, Mazzuoccolo, Zerafa, El.J.Combin. 2021)

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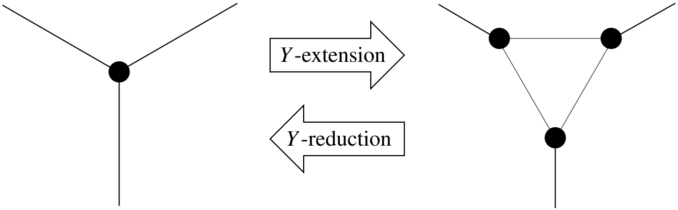

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Proposition (Abreu, Gauci, DL, Mazzuoccolo, Zerafa, El.J.Combin. 2021)

Let G be a hypohamiltonian cubic graph of odd size. Let G' be a graph obtained by performing a Y–extension to all vertices of G except one. Then, $circ(G') = |V(G')| - 1$ and $L(G')$ is not PMH.

Line graphs of graphs with small maximum degree: Final Remarks and Problems

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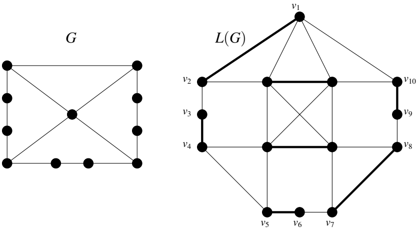

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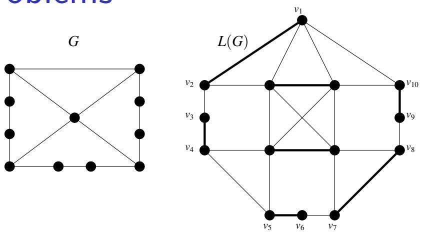

- Recall that the graph in this figure is hamiltonian, but not every perfect matching in its line graph can be extended to a hamiltonian cycle.

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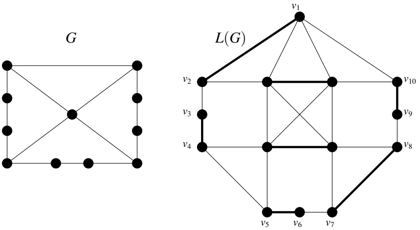

- Recall that the graph in this figure is hamiltonian, but not every perfect matching in its line graph can be extended to a hamiltonian cycle.
- Such example is not regular, and we are not able to find a regular one.

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Problem

Let G be an r–regular hamiltonian graph of even size, for $r \geq 4$. Does $L(G)$ admit the PMH–property?

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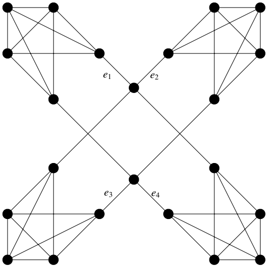

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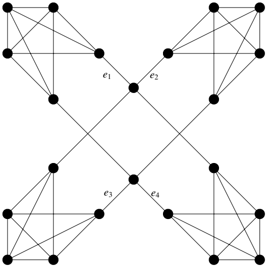
- Note that not all 4–regular (and so not all eulerian) graphs of even size have a PMH line graph.
- A non-hamiltonian example is given in Figure
- It is not hard to check that every perfect matching of $L(G)$ which contains the edges e_1e_2 and e_3e_4 cannot be extended to a hamiltonian cycle of $L(G)$, whose vertices are given the same label as the corresponding edges in G.



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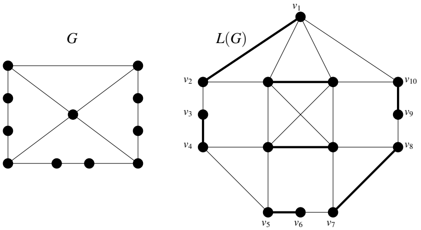

Line graphs of graphs with small maximum degree: Final Remarks and Problems

are both not simultaneously eulerian and hamiltonian, we pose a further problem:

Line graphs of graphs with small maximum degree: Final Remarks and Problems

Problem

Let G be a graph of even size which is both eulerian and hamiltonian. Does $L(G)$ admit the PMH–property?

Other classes of graphs whose line graphs admit the PMH–property: Arbitrarily traceable graphs

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- A graph G is said to be arbitrarily traceable (or equivalently randomly eulerian) from a vertex $v \in V(G)$ if every walk starting from v and not containing any repeated edges can be completed to an eulerian tour.

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Theorem (Abreu, Gauci, DL, Mazzuoccolo, Zerafa, El.J.Combin. 2021)

Let G be an arbitrarily traceable graph from some vertex such that G is of even size. Then, its line graph has the PMH–property.

Other classes of graphs whose line graphs admit the PMH–property: Complete graphs

- In 1976 Daykin proved

Theorem (Daykin - 1976)

If the edges of the complete graph K_n, for $n \geq 6$, are coloured in such a way that no three edges of the same colour are incident to any given vertex, then there exists a properly coloured Hamiltonian cycle.

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Which we have used to prove that

Theorem (Abreu, Gauci, D.L., Mazzuoccolo, Zerafa - El.J.Combin. 2021)

For $n \equiv 0, 1 \mod 4$, $L(K_n)$ is PMH.

Other classes of graphs whose line graphs admit the PMH–property: Complete Bipartite graphs

Theorem (C.C. Chen and D.E. Daykin - 1976)

Consider an edge-colouring of the complete bipartite graph $K_{m,m}$ such that no vertex is incident to more than k edges of the same colour. If $m \geq 25k$, then there exists a properly coloured Hamiltonian cycle.

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So for the case $k = 2$, one could obtain that $L(K_{m,m})$ is PMH for every even $m \geq 50$.

However, using a different and more technical approach the following has been proved:

Theorem (Abreu, J.B. Gauci, J.P. Zerafa - 2021)

Let m_1 be an even integer and let $m_2 \geq 1$. Then, $L(K_{m_1,m_2})$ does not have the PH–property if and only if $m_1 = 2$ and m_2 is odd.

Thank you!