Distance-regular graphs from the Mathieu groups

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Supported by **CSF** (Croatian Science Foundation), project **5713**

Combinatorial Designs and Codes (July 2021, Rijeka, Croatia)
Satellite event of the 8ECM

July 12, 2021

Dean Crnković, Nina Mostarac and Andrea Švob, Distance-regular graphs obtained from the Mathieu groups and new block designs, submitted

- construction of distance-regular graphs admitting a transitive action of the Mathieu groups (five sporadic simple groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24})
- motivation – further contribution to the classification of transitive distance regular graphs (DRGs), especially those admitting a transitive action of a simple group

Distance regular graphs (DRGs)

Association scheme (introduced by Bose and Shimamoto in 1952)

A d-class **association scheme** on a finite non-empty set Ω is an ordered pair $(\Omega, \mathcal{R})$ with $\mathcal{R} = \{R_0, R_1, \ldots, R_d\}$ a set of non-empty relations on Ω, such that the following axioms hold.

1. $\mathcal{R}$ is a partition of Ω^2.
2. R_0 is the identity relation.
3. For every relation $R_i \in \mathcal{R}$, its converse $R_i^T = \{(y, x) : (x, y) \in R_i\}$ is equal to R_i.
4. There are constants p_{ij}^k known as the **intersection numbers** of the association scheme $\mathcal{R}$, such that for $(x, y) \in R_k$, the number of elements z in Ω for which $(x, z) \in R_i$ and $(z, y) \in R_j$ equals p_{ij}^k.

Distance regular graphs (DRGs)

Let Γ be a graph with diameter d, and $\delta(u, v)$ the distance between vertices u and v of Γ.

The *ith-neighborhood* of a vertex v is the set $\Gamma_i(v) = \{w : \delta(v, w) = i\}$.

Let Γ_i be the *ith-distance graph* of Γ, i.e. the graph with the same vertex set as Γ, with adjacency in Γ_i defined by the ith distance relation in Γ.

Distance-regular graph

Graph Γ is **distance-regular** if the distance relations of Γ give the relations of a d-class association scheme, that is, for every choice of $0 \leq i, j, k \leq d$, all vertices v and w with $\delta(v, w) = k$ satisfy $|\Gamma_i(v) \cap \Gamma_j(w)| = p^k_{ij}$ for some constant p^k_{ij}.

- A distance-regular graph Γ is necessarily regular with degree $k = p^0_{11}$.

The sequence of integers $\{b_0, b_1, \ldots, b_{d-1}; c_1, c_2, \ldots, c_d\}$ such that $b_i = p^i_{i+1,1}$ and $c_i = p^i_{i-1,1}$ for $0 \leq i \leq d$, is called the **intersection array of the DRG** Γ (notice that $b_d = c_0 = 0$). Clearly, $b_0 = k$, $c_1 = 1$.

Strongly regular graphs (SRGs)

Strongly regular graph

A regular graph is **strongly regular** with parameters (v, k, λ, μ) if it has v vertices, degree k, and if any two adjacent vertices are together adjacent to λ vertices, while any two non-adjacent vertices are together adjacent to μ vertices.
In that case it is denoted by $\mathrm{SRG}(v, k, \lambda, \mu)$.

Remark

Every **strongly regular** graph with $\mu \neq 0$ is a distance-regular graph with diameter 2 and the **intersection array** given by $\{k, k-1-\lambda; 1, \mu\}$.

Adjacency matrices for orbits of G on $\Omega \times \Omega$

Let G be a finite permutation group acting on the finite set Ω.
This action induces the action of the group G on the set $\Omega \times \Omega$.

If the rank of G is r, then it has r orbits on $\Omega \times \Omega$.

Let $|\Omega| = n$ and Δ_i is one of these orbits. The **adjacency matrix for the orbit Δ_i** is the $n \times n$ matrix A_i, with rows and columns indexed by Ω and entries

$$A_i(\alpha, \beta) = \begin{cases} 1, & \text{if } (\alpha, \beta) \in \Delta_i \\ 0, & \text{otherwise.} \end{cases} .$$

If A_i is symmetric, then the corresponding orbit is called **self-paired**.
If $A_i = A_j^T$, then the corresponding orbits are called **mutually paired**.

Construction

- D. Crnković, V. Mikulić Crnković, A. Švob, On some transitive combinatorial structures constructed from the unitary group $U(3,3)$, J. Statist. Plann. Inference 144 (2014), 19–40.

Construction

Let G be a finite permutation group acting transitively on the set Ω and $A_0, \dots, A_d$ be the adjacency matrices for orbits of G on $\Omega \times \Omega$. Let $\{B_1, \dots, B_t\} \subseteq \{A_1, \dots, A_d\}$ be a set of adjacency matrices for the self-paired or mutually paired orbits. Then $M = \sum_{i=1}^{t} B_i$ is the adjacency matrix of a **regular graph** Γ. The group G acts **transitively** on the set of vertices of the graph Γ.

Using this method one can construct **all regular graphs** admitting a **transitive** action of the group G.

Construction from Mathieu groups

Mathieu groups: the five sporadic simple groups M_{11}, M_{12}, M_{22}, M_{23} and M_{24}

- multiply transitive permutation groups on 11, 12, 22, 23 or 24 objects, respectively

We constructed and partially classified DRGs (SRGs and DRGs of diameter $d \geq 3$) from the Mathieu groups:

- up to 2000 vertices and for which the rank of the permutation representation of the group is at most 25 (i.e. the number of orbits of the stabiliser acting on the cosets is at most 25) admitting a transitive action of the group M_{11},
- up to 2000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group M_{12},
- up to 2000 vertices and for which the rank of the permutation representation of the group is at most 30 admitting a transitive action of the group M_{22},
- up to 10000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group M_{23},
- up to 10000 vertices and for which the rank of the permutation representation of the group is at most 20 admitting a transitive action of the group M_{24}.

The Mathieu group M_{11}

M_{11} is of order 7920. Only 19 of all 39 conjugacy classes of subgroups lead to a permutation representation of rank at most 25 and index at most 2000.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^1	S_5	144	55	3	yes
H_2^1	$Z_9 : QD16$	120	66	4	yes
H_3^1	$Z_{11} : Z_5$	55	144	6	no
H_4^1	$GL(2,3)$	48	165	8	yes
H_5^1	$S_3 \times S_3$	36	220	16	no
H_6^1	S_4	24	330	23	no

Table: Subgroups of the group M_{11}

Theorem

Up to isomorphism there are exactly **five** SRGs and exactly **three** DRGs of diameter $d \geq 3$ with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 25, **admitting a transitive action of the group M_{11}**. The SRGs have parameters $(55, 18, 9, 4)$, $(66, 20, 10, 4)$, $(144, 55, 22, 20)$, $(144, 66, 30, 30)$ and $(330, 63, 24, 9)$, and the DRGs have 165, 220 and 330 vertices, respectively. Details about the obtained SRGs and DRGs with $d \geq 3$ are given in tables below.

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma^1_1 = \Gamma(M_{11}, H^1_1)$	(55,18,9,4)	S_{11}
$\Gamma^1_2 = \Gamma(M_{11}, H^1_2)$	(66,20,10,4)	S_{12}
$\Gamma^1_3 = \Gamma(M_{11}, H^1_3)$	(144,55,22,20)	M_{11}
$\Gamma^1_4 = \Gamma(M_{11}, H^1_3)$	(144,66,30,30)	$M_{12} : Z_2$
$\Gamma^1_5 = \Gamma(M_{11}, H^1_6)$	(330,63,24,9)	S_{11}

Table: SRGs constructed from the group M_{11}

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma^1_6 = \Gamma(M_{11}, H^1_4)$	165	3	$\{24, 14, 6; 1, 4, 9\}$	S_{11}
$\Gamma^1_7 = \Gamma(M_{11}, H^1_5)$	220	3	$\{27, 16, 7; 1, 4, 9\}$	S_{12}
$\Gamma^1_8 = \Gamma(M_{11}, H^1_6)$	330	4	$\{28, 18, 10, 4; 1, 4, 9, 16\}$	S_{11}

Table: DRGs constructed from the group M_{11}, $d \geq 3$

Remark

- All SRGs given in Table are known before.
- The graphs Γ^1_6, Γ^1_7 and Γ^1_8 are unique graphs with the given intersection arrays, known as Johnson graphs, $J(11, 3)$, $J(12, 3)$ and $J(11, 4)$, respectively.

The Mathieu group M_{12}

M_{12} has order 95040 and up to conjugation has 147 subgroups. Only 31 conjugacy classes of subgroups lead to a permutation representation of rank at most 20 and of index at most 2000.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^2	$(A_6.Z_2):Z_2$	1440	66	3	yes
H_2^2	$L(2,11)$	660	144	5	no
H_3^2	$((E_9:Q_8):Z_3):Z_2$	432	220	5	yes
H_4^2	$((E_8:E_4):Z_3):Z_2$	192	495	11	yes
H_5^2	S_5	120	792	15	no

Table: Subgroups of the group M_{12}

Theorem

Up to isomorphism there are exactly **seven** SRGs and exactly **three** DRGs of diameter $d \geq 3$ with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group M_{12}. The SRGs have parameters $(66, 20, 10, 4)$, $(144, 66, 30, 30)$, $(144, 55, 22, 20)$, $(144, 22, 10, 2)$ and $(495, 238, 109, 119)$, and the DRGs have 220, 495 and 792 vertices, respectively. Details about the obtained SRGs and DRGs with $d \geq 3$ are given in tables below.

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^2 = \Gamma(M_{12}, H_1^2)$	(66,20,10,4)	S_{12}
$\Gamma_2^2 = \Gamma(M_{12}, H_2^2)$	(144,66,30,30)	$M_{12} : Z_2$
$\Gamma_3^2 = \Gamma(M_{12}, H_2^2)$	(144,66,30,30)	$M_{12} : Z_2$
$\Gamma_4^2 = \Gamma(M_{12}, H_2^2)$	(144,66,30,30)	M_{12}
$\Gamma_5^2 = \Gamma(M_{12}, H_2^2)$	(144,55,22,20)	$M_{12} : Z_2$
$\Gamma_6^2 = \Gamma(M_{12}, H_2^2)$	(144,22,10,2)	$S_{12} \wr S_2$
$\Gamma_7^2 = \Gamma(M_{12}, H_4^2)$	(495,238,109,119)	$O^-(10,2) : Z_2$

Table: SRGs constructed from the group M_{12}

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_8^2 = \Gamma(M_{12}, H_3^2)$	220	3	$\{27, 16, 7; 1, 4, 9\}$	S_{12}
$\Gamma_9^2 = \Gamma(M_{12}, H_4^2)$	495	4	$\{32, 21, 12, 5; 1, 4, 9, 16\}$	S_{12}
$\Gamma_{10}^2 = \Gamma(M_{12}, H_5^2)$	792	5	$\{35, 24, 15, 8, 3; 1, 4, 9, 16, 25\}$	S_{12}

Table: DRGs constructed from the group M_{12}, $d \geq 3$

Remark

Γ_1^2 is isomorphic to the triangular graph $T(12)$. The adjacency matrices of non-isomorphic SRGs Γ_2^2, Γ_3^2 and Γ_4^2 are the incidence matrices of symmetric designs with parameters $(144, 66, 30)$, designs with Menon parameters (related to a regular Hadamard matrix of order 144). These symmetric designs have been described in:

- W. Lempken, Two new symmetric 2-(144,66,30) designs, preprint, 1999.
- W. Wirth, Konstruktion symmetrischer Designs, PhD thesis, Johannes Gutenberg-Universität Mainz, 2000.

Remark

According to Brouwer's table, known graphs with the parameters equal to the parameters of Γ_5^2 (not isomorphic to Γ_3^1) are obtainable from orthogonal arrays $OA(12, 5)$. Our graph is **new**.
Γ_6^2 is unique graph with the given parameters and Γ_7^2 is isomorphic to the $O^-(10, 2)$ polar graph.

Remark

Γ_8^2, Γ_9^2 *and* Γ_{10}^2 *are unique graphs with the given intersection arrays, known as Johnson graphs,* $J(12, 3)$, $J(12, 4)$ *and* $J(12, 5)$*, respectively.*

- A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

The Mathieu group M_{22}

M_{22} is of order 443520. Only 21 of total 156 conjugacy classes of subgroups lead to a permutation representation of rank at most 30 and index at most 2000.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^3	$E_{16} : A_6$	5760	77	3	yes
H_2^3	A_7	2520	176	3	yes
H_3^3	$E_{16} : S_5$	1920	231	4	yes
H_4^3	$E_8 : L(3,2)$	1344	330	5	yes
H_5^3	$L(2,11)$	660	672	6	yes
H_6^3	$(A_4 \times A_4) : Z_2$	288	1540	22	no

Table: Subgroups of the group M_{22}

Theorem

Up to isomorphism there are exactly **five** strongly regular graphs and exactly **three** distance-regular graphs of diameter $d \geq 3$ with at most 2000 vertices and for which the rank of the permutation representation of the group is at most 30, admitting a transitive action of the group M_{22}. The SRGs have parameters $(77, 16, 0, 4)$, $(176, 70, 18, 34)$, $(231, 30, 9, 3)$, $(231, 40, 20, 4)$ and $(672, 176, 40, 48)$, and the DRGs have 330, 672 and 1540 vertices, respectively. Details about the obtained SRGs and DRGs with $d \geq 3$ are given in tables below.

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^3 = \Gamma(M_{22}, H_1^3)$	$(77, 16, 0, 4)$	$M_{22} : Z_2$
$\Gamma_2^3 = \Gamma(M_{22}, H_2^3)$	$(176, 70, 18, 34)$	M_{22}
$\Gamma_3^3 = \Gamma(M_{22}, H_3^3)$	$(231, 30, 9, 3)$	$M_{22} : Z_2$
$\Gamma_4^3 = \Gamma(M_{22}, H_3^3)$	$(231, 40, 20, 4)$	S_{22}
$\Gamma_5^3 = \Gamma(M_{22}, H_5^3)$	$(672, 176, 40, 48)$	$(U(6,2) : Z_2) : Z_2$

Table: SRGs constructed from the group M_{22}

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_6^3 = \Gamma(M_{22}, H_4^3)$	330	4	$\{7, 6, 4, 4; 1, 1, 1, 6\}$	$M_{22} : Z_2$
$\Gamma_7^3 = \Gamma(M_{22}, H_5^3)$	672	3	$\{110, 81, 12; 1, 18, 90\}$	$M_{22} : Z_2$
$\Gamma_8^3 = \Gamma(M_{22}, H_6^3)$	1540	3	$\{57, 36, 17; 1, 4, 9\}$	S_{22}

Table: DRGs constructed from the group M_{22}, $d \geq 3$

- Γ_1^3 and Γ_2^3 are unique graphs with these parameters. Γ_3^3 is isomorphic to SRG known as the Cameron graph. Γ_4^3 is isomorphic to the triangular graph $T(22)$ and Γ_5^3 to the $U(6,2)$-graph.
- Γ_6^3 is isomorphic to doubly truncated Witt graph known as M_{22}-graph. Γ_7^3 is isomorphic to DRG constructed by Soicher (so far the only known DRG with this intersection array). Γ_8^3 is the Johnson graph $J(22, 3)$.

L. H Soicher, Yet another distance-regular graph related to Golay code, Electronic J. Combin. 2 (1995), N1, 4pp.

The Mathieu group M_{23}

M_{23} has order 10200960. Only 14 of all 204 conjugacy classes of subgroups lead to a permutation representation of rank at most 20 and of index at most 10000.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^4	$L(3,4):Z_2$	40320	253	3	yes
H_2^4	$E_{16}:A_7$	40320	253	3	yes
H_3^4	A_8	20160	506	4	yes
H_4^4	M_{11}	7920	1288	4	yes
H_5^4	$E_{16}:(A_5:S_3)$	5760	1771	8	yes

Table: Subgroups of the group M_{23}

Theorem

Up to isomorphism there are exactly **three** strongly regular graphs and exactly **two** distance-regular graphs of diameter $d \geq 3$ with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group M_{23}. The SRGs have parameters $(253, 42, 21, 4)$, $(253, 112, 36, 60)$ and $(1288, 495, 206, 180)$, and the DRGs have 506 and 1771 vertices, respectively. Details about the obtained SRGs and DRGs with $d \geq 3$ are given in tables below.

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^4 = \Gamma(M_{23}, H_1^4)$	(253,42,21,4)	S_{23}
$\Gamma_2^4 = \Gamma(M_{23}, H_2^4)$	(253,112,36,60)	M_{23}
$\Gamma_3^4 = \Gamma(M_{23}, H_4^4)$	(1288,495,206,180)	M_{24}

Table: SRGs constructed from the group M_{23}

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_4^4 = \Gamma(M_{23}, H_3^4)$	506	3	$\{15, 14, 12; 1, 1, 9\}$	M_{23}
$\Gamma_5^4 = \Gamma(M_{23}, H_5^4)$	1771	3	$\{60, 38, 18; 1, 4, 9\}$	S_{23}

Table: DRG constructed from the group M_{23}, $d \geq 3$

- Γ_1^4 is isomorphic to the triangular graph $T(23)$. Γ_2^4 can be constructed from M_{23} as a rank 3 graph, and Γ_3^4 (isomorphic to Γ_2^5) can be constructed from M_{24} as a rank 3 graph.
- Γ_4^4 is isomorphic to the distance-regular graph that can be obtained from residual design of Steiner system $S(5, 8, 24)$. Γ_5^4 is known as Johnson graph $J(23, 3)$.

The Mathieu group M_{24}

M_{24} is of order 244823040. Only 15 of all 1529 conjugacy classes of subgroups lead to a permutation representation of rank at most 20 and index at most 10000.

Subgroup	Structure	Order	Index	Rank	Primitive
H_1^5	$M_{22}:Z_2$	887040	276	3	yes
H_2^5	$E_{16}:A_8$	322560	759	4	yes
H_3^5	$M_{12}:Z_2$	190080	1288	3	yes
H_4^5	$(L(3,4):Z_3):Z_2$	120960	2024	5	yes

Table: Subgroups of the group M_{24}

Theorem

Up to isomorphism there are exactly **two** strongly regular graphs and exactly **two** distance-regular graphs of diameter $d \geq 3$ with at most 10000 vertices and for which the rank of the permutation representation of the group is at most 20, admitting a transitive action of the group M_{24}. The SRGs have parameters $(276, 44, 22, 4)$ and $(1288, 495, 206, 180)$, and the DRGs have 759 and 2024 vertices, respectively. Details about the obtained SRGs and DRGs with $d \geq 3$ are given in tables below.

Graph Γ	Parameters	$Aut(\Gamma)$
$\Gamma_1^5 = \Gamma(M_{24}, H_1^5)$	$(276, 44, 22, 4)$	S_{24}
$\Gamma_2^5 = \Gamma(M_{24}, H_3^5)$	$(1288, 495, 206, 180)$	M_{24}

Table: SRGs constructed from the group M_{24}

Graph Γ	Number of vertices	Diameter	Intersection array	$Aut(\Gamma)$
$\Gamma_3^5 = \Gamma(M_{24}, H_2^5)$	759	3	$\{30, 28, 24; 1, 3, 15\}$	M_{24}
$\Gamma_4^5 = \Gamma(M_{24}, H_4^5)$	2024	3	$\{63, 40, 19; 1, 4, 9\}$	S_{24}

Table: DRGs constructed from the group M_{24}, $d \geq 3$

- Γ_1^5 is isomorphic to the triangular graph $T(24)$. Γ_2^5 (isomorphic to Γ_3^4) can be constructed from M_{24} as a rank 3 graph.
- Γ_3^5 is unique distance-regular graph known as near hexagon which can be obtained from Steiner system $S(5, 8, 24)$. Γ_4^5 is known as Johnson graph $J(24, 3)$.

Codes and block designs

Let w_i denote the number of codewords of weight i in a code C of length n.

The **weight distribution** of C is the list $[\langle i, w_i : 0 \leq i \leq n \rangle]$.

The **support** of a nonzero vector $x = (x_1, \dots, x_n) \in F_q^n$ is the set of indices of its nonzero coordinates, i.e. $\mathrm{supp}(x) = \{i | x_i \neq 0\}$.

The **support design** of a code of length n **for a given nonzero weight** w is the design with points the n coordinate indices and blocks the supports of all codewords of weight w.

Block designs

We describe block designs obtained from the code $[176, 22, 50]_2$ spanned by the adjacency matrix of the graph Γ_2^3, SRG(176,70,18,34).

- From the supports of all codewords of the weights of the code $[176, 22, 50]_2$ we obtain block designs on 176 points on which the finite simple group Higman-Sims acts as the automorphism group.

The weight distribution of the code $[176, 22, 50]_2$ is:

$[\langle 0,1\rangle, \langle 50,176\rangle, \langle 56,1100\rangle, \langle 64,4125\rangle, \langle 66,5600\rangle, \langle 70,17600\rangle, \langle 72,15400\rangle, \langle 78,193600\rangle,$
$\langle 80,604450\rangle, \langle 82,462000\rangle, \langle 86,369600\rangle, \langle 88,847000\rangle, \langle 90,369600\rangle, \langle 94,462000\rangle, \langle 96,604450\rangle,$
$\langle 98,193600\rangle, \langle 104,15400\rangle, \langle 106,17600\rangle, \langle 110,5600\rangle, \langle 112,4125\rangle, \langle 120,1100\rangle, \langle 126,176\rangle, \langle 176,1\rangle]$

Block design D	Parameters (v,k,λ)	$Aut(D)$	Block design D	Parameters (v,k,λ)	$Aut(D)$
D_1	(176,50,14), b=176	HS	D_7	(176,78,37752), b=193600	HS
D_2	(176,56,110), b=1100	HS	D_8	(176,80,124030), b=604450	HS
D_3	(176,64,540), b=4125	HS	D_9	(176,82,99630), b=462000	HS
D_4	(176,66,780), b=5600	HS	D_{10}	(176,86,87720), b=369600	HS
D_5	(176,70,2760), b=17600	HS	D_{11}	(176,88,210540), b=847000	HS
D_6	(176,72,2556), b=15400	HS			

Table: Block designs from the code $[176, 22, 50]_2$

In Table we did not include the support designs obtained from the weights 90–176 since they give rise to the complements of the block designs given in Table.

The action of the HS group on 176 points is 2-homogeneous. It is well known that if an automorphism group G of a binary code is t-homogeneous, then the codewords of any given weight at least t hold t-designs admitting an action of G.

New block designs from the code $[176, 22, 50]_2$

- The Mathieu group M_{22} is a subgroup of the Higman–Sims group.
- M_{22} as a subgroup of the HS group does not act 2-homogeneously on 176 points.
- From each of the constructed block designs we constructed structures obtained from the M_{22}-orbits on the blocks.

Block design	Parameters (v, k, λ)	Aut. group
D_5	(176,70,414), b=2640	M_{22}
D_5	(176,70,966), b=6160	M_{22}
D_8	(176,80,5688), b=27720	M_{22}
D_8	(176,80,15168), b=73920	M_{22}
D_8	(176,80,15168), b=73920	M_{22}
D_8	(176,80,22752), b=110880	M_{22}
D_{10}	(176,86,13158), b=55440	M_{22}
D_{10}	(176,86,17544), b=73920	M_{22}

Table: New block designs from the code $[176, 22, 50]_2$

- These 8 block designs are new, up to our best knowledge.

PD-sets for codes from obtained DRGs

Algorithm of **permutation decoding** (introduced in 1964 by MacWilliams) uses sets of code automorphisms called **PD-sets**, defined **with respect to a given information set** of the code.

Information set

Let $C \subseteq \mathbb{F}_p^n$ be a linear $[n,k,d]$ code. For $I \subseteq \{1,\dots,n\}$ let $p_I : \mathbb{F}_p^n \to \mathbb{F}_p^{|I|}$, $x \mapsto x|_I$, be an I-projection of $\mathbb{F}_p^n$. Then I is called an **information set** for C if $|I| = k$ and $p_I(C) = \mathbb{F}_p^{|I|}$.

- The set of the first k coordinates for a code with a generating matrix in the standard form is an **information set**. The first k coordinates are then called *information symbols* and the last $n-k$ coordinates are the *check symbols* and they form the corresponding **check set**.

PD-set

Let $C \subseteq \mathbb{F}_p^n$ be a linear $[n,k,d]$ code that can correct at most t errors, and let I be an information set for C. A subset $S \subseteq \mathrm{Aut}C$ is a PD-set for C if every t-set of coordinate positions can be moved by at least one element of S out of the information set I.

PD-sets for codes from obtained DRGs

For any of the constructed DRG Γ^i_j, let C^i_j denote the **linear code spanned by the adjacency matrix of Γ^i_j over $\mathbb{F}_p$.**

Let g denote the Gordon bound for the size of the PD-set of a code (theoretical lower bound given by Gordon in 1982).

Code C	Parameters $[n, k, d]$	$Aut(C)$	t	g	Size of PD-set
C^1_1	[55,10,10]	S_{11}	4	5	5
C^1_2, C^2_1	[66,10,20], so	S_{12}	9	15	55
C^1_5	[330,286,6]	S_{11}	2	60	420
C^1_6	[165,120,4]	S_{11}	1	4	5
C^1_8	[330,120,8]	S_{11}	3	7	22
C^3_1	[77,20,16],so	$M_{22}:Z_2$	7	19	110
C^4_5	[1771,1540,4]	S_{23}	1	8	23

Table: PD-sets for codes from constructed DRGs from Mathieu groups

Code C^1_2 is equivalent to the code C^2_1.

Thank you!