Small complete caps in $\mathrm{PG}(4n+1, q)$

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complete caps are 1-saturating sets

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parity check matrix of $\mathcal{C}$: generator matrix of $\mathcal{C}^\perp$

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1-*saturating set* $\mathcal{X}$ *of* $\mathrm{PG}(r-1,q)$ *of size* s	$\Longleftrightarrow$	*columns of a parity check matrix* H *of an* $[s, s-r]_q$ *code* $\mathcal{C}$ *with covering radius* 2

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- 11*-complete cap of* $\mathrm{PG}(4,3) \to [11,6]_3$ *ternary Golay code*

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trivial lower bound: $t_2(r-1, q) \geq \sqrt{2}q^{\frac{r-2}{2}}$

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Kim & Vu 2003

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Pambianco & Storme 1996 $\mathrm{PG}(r-1, q)$, q *even*

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$$\frac{5}{2}q^{\frac{r-2}{2}} + 3\left(q^{\frac{r-4}{2}} + \cdots + q\right) + 2, \ r \text{ even}$$

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Özbudak, *On maximal curves and linearized permutation polynomials over finite fields*, 2001.

A geometric description

$(2n+1) \times (2n+1)$ symmetric matrices over $\mathbb{F}_{q^{2n+1}}$

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\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
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vector space of dimension $(n+1)(2n+1)$ over $\mathbb{F}_q$

A geometric description

$(2n+1)\times(2n+1)$ symmetric matrices over $\mathbb{F}_{q^{2n+1}}$

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\vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
a_{n-1} & \dots & a_0^{q^{n-1}} & a_1^{q^{n-1}} & a_2^{q^{n-1}} & a_3^{q^{n-1}} & \dots & a_n^{q^{2n}}\\
a_n & \dots & a_1^{q^{n-1}} & a_0^{q^n} & a_1^{q^n} & a_2^{q^n} & \dots & a_n^{q^n}\\
a_n^{q^{n+1}} & \dots & a_2^{q^{n-1}} & a_1^{q^n} & a_0^{q^{n+1}} & a_1^{q^{n+1}} & \dots & a_{n-1}^{q^{n+1}}\\
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$$\mathrm{PG}(W)\simeq\mathrm{PG}(n(2n+3),q)$$

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Veronese variety of $\mathrm{PG}(W)$: locus of the zeros of all determinants of 2×2 submatrices of $M(a_0, \ldots, a_n)$

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THANK YOU