New methods to attack the Buratti-Horak-Rosa Conjecture

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Notation

- K_v denotes the complete graph whose vertex-set is $\{0, 1, \ldots, v-1\}$ for any positive integer v.
- *length* $\ell(x, y)$ of an edge $[x, y]$ of K_v as

$$\ell(x, y) = \min(|x-y|, v-|x-y|).$$

- Given a subgraph Γ of K_v, the list of edge-lengths of Γ is the list $\ell(\Gamma)$ of the lengths (taken with their respective multiplicities) of all the edges of Γ. We write $L = \{1^{a_1}, 2^{a_2}, \ldots, t^{a_t}\}$ and $|L| = \sum a_i$

Buratti's Conjecture - a combinatorial disease

Conjecture (M. Buratti, 2007)

For any prime $p = 2n + 1$ and any list L of $2n$ positive integers not exceeding n, there exists a Hamiltonian path H of K_p with $\ell(H) = L$.

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- Easily seen to be true if L has just one edge-length
- If L has two distinct edge lengths, then solved by Dinitz and Janiszewski (2009) and Horak and Rosa (2009)
- Meszka showed true for all primes $p \leq 23$

Buratti-Horak-Rosa (BHR) Conjecture

Conjecture (P. Horak and A. Rosa)

Let L be a list of $v-1$ positive integers not exceeding $\left\lfloor \frac{v}{2} \right\rfloor$. Then there exists a Hamiltonian path H of K_v such that $\ell(H) = L$ if, and only if, the following condition holds:

$$\begin{array}{c}\textit{for any divisor } d \textit{ of } v\textit{, the number of multiples of } d \\ \textit{appearing in } L \textit{ does not exceed } v-d.\end{array} \tag{1}$$

Buratti-Horak-Rosa (BHR) Conjecture

BHR Conjecture is true when:

- $v \leq 18$, Meszka by computer
- three distinct edge lengths of $1, 2, 3$ by Capparelli and Del Fra (2010)
- three distinct edge lengths of $1, 2, 5$ or $1, 3, 5$ or $2, 3, 5$ by Pasotti and Pellegrini (2014)
- three distinct edge lengths of $1, 2, 4$ or $1, 2, 6$ or $1, 2, 8$ by Pasotti and Pellegrini (2014)
- $L = \{1^a, 2^b, t^c\}$ with t even and $a + b \geq t - 1$ by Pasotti and Pellegrini (2014)
- $1, 2, 3, 5$ by Pasotti and Pellegrini (2014)

Cyclic realizations

A *cyclic realization* of a list L with $v-1$ elements each from the set $\{1,\dots,\lfloor\frac{v}{2}\rfloor\}$ is a Hamiltonian path $[x_0,x_1,\dots,x_{v-1}]$ of K_v such that the list of edge-lengths $\{\ell(x_i,x_{i+1}) \mid i=0,\dots,v-2\}$ equals L.

$\mathrm{BHR}(L)$ can be reformulated:
every such a list L has a cyclic realization if and only if condition (1) is satisfied. For example, the path $[0,5,11,3,9,1,8,2,10,4,12,6,7]$ is a cyclic realization of $\{1,5^5,6^6\}$.

Now, let L be a list with $v-1$ positive integers not exceeding $v-1$. A *linear realization* of L, denoted by rL, is a Hamiltonian path $[x_0, x_1, \ldots, x_{v-1}]$ of K_v such that $L = \{|x_i - x_{i+1}| \mid i = 0, \ldots, v-2\}$. By *standard* linear realization, we mean a linear realization starting with 0. For instance, one can easily check that the path $[0, 4, 1, 5, 2, 3, 7, 11, 8, 12, 9, 6, 10, 13, 17, 14, 18, 15, 16, 19]$ is a standard linear realization of $\{1^2, 3^9, 4^8\}$

Remark

Every linear realization of a list L can be viewed as a cyclic realization of a suitable list L' but not necessarily of the same list. For example, the path $[0, 1, 2, 7, 3, 4, 5, 6]$ is a linear realization of $L = \{1^5, 4, 5\}$ and a cyclic realization of $L' = \{1^5, 3, 4\}$. If all the elements in the list are less than or equal to $\left\lfloor \frac{|L|+1}{2} \right\rfloor$, then every linear realization of L is also a cyclic realization of the same list L.

Let $\boldsymbol{g} = [g_1, g_2, \ldots, g_v]$ be a linear realization of a list L. The *reverse*,

$$\mathrm{rev}(\boldsymbol{g}) = [g_v, g_{v-1}, \ldots, g_1]$$

is also a linear realization of L, as is its *complement*,

$$\bar{\boldsymbol{g}} = [v - 1 - g_1, v - 1 - g_2, \ldots, v - 1 - g_v].$$

The *translation* of $\boldsymbol{g}$ by a is $\boldsymbol{g} + a = [g_1 + a, \ldots, g_v + a]$, which has the same absolute differences as $\boldsymbol{g}$.

Concatenation

Theorem

Let L_1 and L_2 be lists. If each of L_1 and L_2 has a standard linear realization, then $L = L_1 \cup L_2$ has a linear realization.

Proof.

Let $\boldsymbol{g} = [g_1, \ldots, g_m]$ and $\boldsymbol{h} = [h_1, \ldots, h_n]$ be linear realizations of L_1 and L_2 respectively with $g_1 = 0 = h_1$. Consider the sequence $\boldsymbol{g} \oplus \boldsymbol{h}$ obtained by concatenating $\mathrm{rev}(\bar{\boldsymbol{g}})$ and $[h_2, \ldots, h_n] + (m-1)$, which has length $m + n - 1$.
The first $m-1$ absolute differences give the elements of L_1 and, as the mth element of the sequence is $m - 1 = h_1 + m - 1$, the remaining $n-1$ absolute differences give the elements of L_2.
Hence the sequence is a linear realization of L. $\square$

Insertion

Definition

Let x be a positive integer. We say that a linear realization rL of a list L is:

- of type $\mathcal{A}_x$ if the two vertices $|L|-x$ and $|L|-x+1$ are adjacent in rL;
- of type $\mathcal{B}_x$ if the two vertices $|L|-x$ and $|L|$ are adjacent in rL.

We define a function η_x acting on linear realizations of type $\mathcal{A}_x$ as follows: the path $\eta_x(rL)$ is obtained from rL by inserting $|L|+1$ between the adjacent vertices $|L|-x$ and $|L|-x+1$. It is easy to see that $\eta_x(rL)$ is a linear realization of $(L \setminus \{1\}) \cup \{x, x+1\}$. Analogously, we define a function μ_x acting on linear realizations of type $\mathcal{B}_x$ in the following way: the path $\mu_x(rL)$ is obtained from rL by inserting $|L|+1$ between the adjacent vertices $|L|-x$ and $|L|$. Note that $\mu_x(rL)$ is a linear realization of $(L \setminus \{x\}) \cup \{1, x+1\}$.

Our results

Theorem

[Ollis, Pasotti, Pellegrini, S. (2021)] Let $x \geq 3$ *be an integer and let* $L = \{1^a, x^b, (x+1)^c\}$ *be an admissible list. Then* $\mathrm{BHR}(L)$ *holds in each of the following cases:*

(1) x *is odd and* $a \geq \min\{3x-3, b+2x-3\}$;

(2) x *is odd,* $a \geq 2x-2$ *and* $c \geq \frac{4}{3}b$;

(3) x *is even and* $a \geq \min\{3x-1, c+2x-1\}$;

(4) x *is even,* $a \geq 2x-1$ *and* $b \geq c$.

Our results

Theorem

[Ollis, Pasotti, Pellegrini, S. (2021)] Let $x \geq 2$ and $c \geq 1$. Let

$$L = \left\{1^a, 2^{b_2}, 4^{b_4}, 6^{b_6}, \ldots, \ell^{b_\ell}, x^c\right\}$$

be an admissible list, where $\ell = 2\left\lfloor \frac{x-1}{2} \right\rfloor$. Then $\mathrm{BHR}(L)$ holds in each of the following cases:

(1) *x is even and $a \geq x - 1$;*

(2) *x is odd and $a \geq 3x - 4$.*

Our results

Theorem

[Ollis, Pasotti, Pellegrini, S. (2021)] Let $L = \{1^a, 2^b, 3^c, 4^d\}$ *be an admissible list, where* $c, d \geq 1$*. Then* $\mathrm{BHR}(L)$ *holds for all* $a \geq 3$ *and all* $b \geq 0$*. Also,* $\mathrm{BHR}(L)$ *holds when* $a = 2$ *and* $b \geq 1$*.*

Theorem

[Ollis, Pasotti, Pellegrini, S. (2021)] Let $L = \{1^{a_1}, \ldots, m^{a_m}\}$ *be an admissible list. Then* $\mathrm{BHR}(L)$ *holds whenever* $a_1 \geq a_2 \geq \cdots \geq a_m > 0$*.*

Thanks!