Erdős-Ko-Rado, Cameron-Liebler and Hilton-Milner results in finite projective spaces

Leo Storme

Ghent University
Dept. of Mathematics: Analysis, Logic and Discrete Mathematics
9000 Ghent
Belgium

Combinatorial Designs and Codes, July 2021

OUTLINE

UNIVERSITEIT GENT

Algebraic combinatorics

Algebraic combinatorics/Association schemes greatly help finite geometry.

Point/k-space incidence matrix

- Π_k = set of k-subspaces in PG(n, q), for $0 \leq k \leq n$.
- $A = (a_{ij})$ = incidence matrix of points and k-spaces of PG(n, q): rows of A are indexed by points p_i and columns by k-spaces PG$(k, q)_j$.

$$a_{ij} = \begin{cases} 1 \Leftrightarrow p_i \in \mathrm{PG}(k, q)_j \\ 0 \Leftrightarrow p_i \notin \mathrm{PG}(k, q)_j \end{cases}$$

DISTANCE-i-RELATION

- $R_i = \{(\pi, \pi') | \pi, \pi' \in \Pi_k, \dim(\pi \cap \pi') = k - i\}, 0 \leq i \leq k + 1$.
- A_i = incidence matrix of relation R_i.
- Relations $R_0, R_1, \ldots, R_{k+1}$ = Grassmann association scheme $J_q(n+1, k+1)$.
- $A_0 = I$ = identity matrix.
- $\sum_{i=0}^{k+1} A_i = J$ = all-one matrix.

DISTANCE-i-RELATION

- $R_i = \{(\pi, \pi') | \pi, \pi' \in \Pi_k, \dim(\pi \cap \pi') = k - i\}, 0 \leq i \leq k+1$.
- A_i = incidence matrix of relation R_i.
- Disjointness matrix A_{k+1} also denoted by K.
  Corresponding graph is a Kneser graph.
- Relations $R_0, R_1, \ldots, R_{k+1}$ = Grassmann association scheme $J_q(n+1, k+1)$.
- j = all-one vector.

DISTANCE-i-RELATION

- $R_i = \{(\pi, \pi')|\pi, \pi' \in \Pi_k, \dim(\pi \cap \pi') = k - i\}, 0 \leq i \leq k + 1$.
- A_i = incidence matrix of relation R_i.
- Matrices A_i symmetrical: real eigenvalues and diagonalizable.

THEOREM

There is orthogonal decomposition $V_0 \perp V_1 \perp \cdots \perp V_{k+1}$ of $\mathbb{R}^{\Pi_k}$ in common eigenspaces of $A_0, A_1, \ldots, A_{k+1}$.
Here $V_0 = \langle j \rangle$.

GAUSSIAN BINOMIAL COEFFICIENT

- *Gaussian binomial coefficient* $\begin{bmatrix} a \\ b \end{bmatrix}_q$ for $a, b \in \mathbb{N}$ and prime power $q \geq 2$ =

$$\begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{(q^a - 1) \cdots (q^{a-b+1} - 1)}{(q^b - 1) \cdots (q - 1)}.$$

- $\begin{bmatrix} a \\ b \end{bmatrix}_q$ = number of b-spaces of the vector space $\mathbb{F}_q^a$ = number of $(b-1)$-spaces in the projective space $\mathrm{PG}(a-1, q)$.
- If field size q is clear from context, we write $\begin{bmatrix} a \\ b \end{bmatrix}$.

EIGENVALUES

- $R_i = \{(\pi, \pi') | \pi, \pi' \in \Pi_k, \dim(\pi \cap \pi') = k - i\}, 0 \leq i \leq k + 1$.
- A_i = incidence matrix of relation R_i.
- $\mathbb{R}^{\Pi_k} = V_0 \perp V_1 \perp \cdots \perp V_{k+1}$.

LEMMA (DELSARTE)

Consider $J_q(n+1, k+1)$. Eigenvalue P_{ji} of distance-i relation for V_j is:

$$P_{ji} = \sum_{s=\max(0,j-i)}^{\min(j,k+1-i)} (-1)^{j+s} \begin{bmatrix} j \\ s \end{bmatrix} \begin{bmatrix} n-k+s-j \\ n-k-i \end{bmatrix} \begin{bmatrix} k+1-s \\ i \end{bmatrix} q^{i(i+s-j)+\frac{(j-s)(j-s-1)}{2}}.$$

FIRST OBSERVATION

- Π_k = set of k-subspaces in $\mathrm{PG}(n,q)$, for $0 \leq k \leq n$.
- $A = (a_{ij})$ = incidence matrix of points and k-spaces of $\mathrm{PG}(n,q)$: rows of A are indexed by points p_i and columns by k-spaces $\mathrm{PG}(k,q)_j$.

$$a_{ij} = \begin{cases} 1 \Leftrightarrow p_i \in \mathrm{PG}(k,q)_j \\ 0 \Leftrightarrow p_i \notin \mathrm{PG}(k,q)_j \end{cases}$$

THEOREM

$$\mathrm{Im}(A^T) = V_0 \perp V_1$$

Equivalent definitions

Theorem (Blokhuis, De Boeck, D'haeseleer)

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 2k+1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Then following properties are equivalent.

1. $\chi \in \mathrm{Im}(A^T)$.
2. $\chi \in (\ker(A))^{\perp}$.
3. *For every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi)) \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.*

Equivalent definitions

Theorem (Blokhuis, De Boeck, D'haeseleer)

1. *Vector* $v = \chi - x\frac{q^{k+1}-1}{q^{n+1}-1}j$ *is a vector in* V_1.
2. $\chi \in V_0 \perp V_1$.
3. *For given* $i \in \{1, \ldots, k+1\}$ *and given* k*-space* π, *number of elements of* $\mathcal{L}$, *meeting* π *in* $(k-i)$*-space is:*

$$\begin{cases} \left((x-1)\frac{q^{k+1}-1}{q^{k-i+1}-1} + q^i\frac{q^{n-k}-1}{q^i-1}\right) q^{i(i-1)} \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k \\ i \end{bmatrix} & \text{if } \pi \in \mathcal{L} \\ x \begin{bmatrix} n-k-1 \\ i-1 \end{bmatrix} \begin{bmatrix} k+1 \\ i \end{bmatrix} q^{i(i-1)} & \text{if } \pi \notin \mathcal{L} \end{cases}$$

4. *for every pair of conjugate switching sets* $\mathcal{R}$ *and* $\mathcal{R}'$, $|\mathcal{L} \cap \mathcal{R}| = |\mathcal{L} \cap \mathcal{R}'|$.

EQUIVALENT DEFINITIONS

DEFINITION

k-spread of $\mathrm{PG}(n,q)$ is partitioning of $\mathrm{PG}(n,q)$ into k-spaces.

k-Spread in $\mathrm{PG}(n,q)$ only exists if $(k+1)|(n+1)$.

THEOREM (BLOKHUIS, DE BOECK, D'HAESELEER)

If $\mathrm{PG}(n,q)$ *has k-spread, then following properties are equivalent to the previous ones.*

1. $|\mathcal{L}\cap\mathcal{S}| = x$ *for every regular k-spread* $\mathcal{S}$ *in* $\mathrm{PG}(n,q)$.
2. $|\mathcal{L}\cap\mathcal{S}| = x$ *for every k-spread* $\mathcal{S}$ *in* $\mathrm{PG}(n,q)$.

FROM 2. TO 3.

THEOREM

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 2k + 1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Then 2. implies 3.:

2. $\chi \in (\ker(A))^{\perp}$.
3. *For every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi)) \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.*

Preparatory lemmas

Lemma

The number of k-spaces disjoint to a fixed k-space π in $\mathrm{PG}(n,q)$ *equals* $q^{k^2+2k+1}\begin{bmatrix} n-k \\ k+1 \end{bmatrix}$.

Lemma

Let $P \in \mathrm{PG}(n,q) \setminus \pi$.
The number of k-spaces through P disjoint to π in $\mathrm{PG}(n,q)$ *equals* $q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}$.

PREPARATORY LEMMAS

Let $\mathcal{Z}$ be set of all k-subspaces in $\mathrm{PG}(n, q)$ disjoint from π, with characteristic vector $\chi_{\mathcal{Z}}$.

LEMMA

Let $P \in \mathrm{PG}(n, q) \setminus \pi$.
The number of k-spaces through P disjoint to π in $\mathrm{PG}(n, q)$ equals $q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix}$.

So

$$A\chi_{\mathcal{Z}} = q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} (j - v_\pi).$$

PREPARATORY LEMMAS

LEMMA

Let π be a k-dimensional subspace in $\mathrm{PG}(n,q)$ *with χ_π the characteristic vector of the set $\{\pi\}$. Let $\mathcal{Z}$ be the set of all k-subspaces in* $\mathrm{PG}(n,q)$ *disjoint from π with characteristic vector $\chi_\mathcal{Z}$, then*

$$\chi_\mathcal{Z} - q^{k^2+k}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}\left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} j - \chi_\pi\right) \in \ker(A).$$

FROM 2. TO 3.

THEOREM

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 2k+1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Then 2. implies 3.:

2. $\chi \in (\ker(A))^{\perp}$.
3. *For every k-space π, the number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi)) \begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.*

Proof: from 2. to 3.

$\chi \in (\ker(A))^{\perp}$, while

$$\chi_{\mathcal{Z}} - q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} j - \chi_{\pi} \right) \in \ker(A).$$

So

$$\chi \cdot \left(\chi_{\mathcal{Z}} - q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} j - \chi_{\pi} \right) \right) = 0.$$

PROOF: FROM 2. TO 3.

$$\chi \cdot \chi_{\mathcal{Z}} - q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} \chi \cdot j - \chi \cdot \chi_\pi \right) = 0.$$

$$\chi \cdot \chi_{\mathcal{Z}} = q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} \chi \cdot j - \chi \cdot \chi_\pi \right).$$

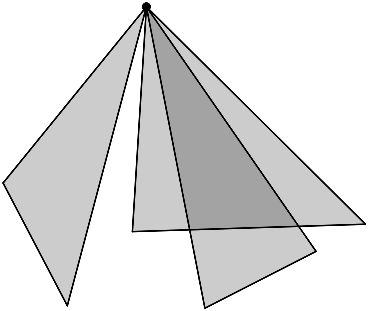
$$|\mathcal{L} \cap \mathcal{Z}| = q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} \left(\begin{bmatrix} n \\ k \end{bmatrix}^{-1} |\mathcal{L}| - \chi(\pi) \right).$$

$$|\mathcal{L} \cap \mathcal{Z}| = q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} (x - \chi(\pi)).$$

Outline

Examples of Cameron-Liebler sets

Example 1: All k-spaces through point P (= point-pencil) = Cameron-Liebler set with parameter $x = 1$.



Examples of Cameron-Liebler sets

Theorem

All k-spaces through point P (= point-pencil) = Cameron-Liebler set with parameter $x = 1$.

Proof: All k-spaces through point P = row of A. □

Examples of Cameron-Liebler sets

Theorem

All k-spaces in hyperplane π = Cameron-Liebler set with parameter $x = \frac{q^{n-k}-1}{q^{k+1}-1}$.

Proof: Equivalent definition 3.

3. For every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi))\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$. □

EXAMPLES OF CAMERON-LIEBLER SETS

- All k-spaces in hyperplane π = Cameron-Liebler set with parameter $x = \frac{q^{n-k}-1}{q^{k+1}-1}$
- **Remark:** $x = \frac{q^{n-k}-1}{q^{k+1}-1} \in \mathbb{Q} \setminus \mathbb{N}$ when $(k+1) \not| (n+1)$.

OUTLINE

CHARACTERIZATION RESULTS

LEMMA

There are no Cameron-Liebler sets of k-spaces in $PG(n, q)$ with parameter $x \in]0, 1[$.

Proof:

- Let $\mathcal{L}$ = Cameron-Liebler set of k-spaces with parameter $x \in]0, 1[$.
- Then $\mathcal{L}$ is not empty set, so suppose $\pi \in \mathcal{L}$.
  By definition 3:

3. For every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi))\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.

- Number of k-spaces of $\mathcal{L}$ skew to π is negative. (Contradiction)

CHARACTERIZATION RESULT FOR $x = 1$

LEMMA

Let $\mathcal{L}$ = Cameron-Liebler set of k-spaces with parameter $x = 1$, then every two k-spaces in $\mathcal{L}$ intersect in at least one point.

Proof: Definition 3.

3. For every k-space π, number of elements of $\mathcal{L}$ disjoint from π is $(x - \chi(\pi))\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$.

- $(x = 1)$ For every k-space $\pi \in \mathcal{L}$, number of elements of $\mathcal{L}$ disjoint from π is $(1 - 1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k} = 0$. □

Conclusion: Cameron-Liebler set of k-spaces with parameter $x = 1$ is set of pairwise intersecting k-spaces.
Link to Erdős-Ko-Rado results.

Erdős-Ko-Rado problem

Problem: What are largest sets of k-sets in n-set, pairwise intersecting in at least one element?

Theorem (Erdős-Ko-Rado)

If S is set of k-sets in n-set Ω, with $2k \leq n$, pairwise intersecting in at least one element, then $|S| \leq \binom{n-1}{k-1}$. If $2k+1 \leq n$, then equality only holds if S consists of all k-sets through fixed element of Ω.

$n = 2k$**:** If $n = 2k$, other sets with equality: all k-sets in fixed subset of size $n-1 = 2k-1$ of Ω.

Erdős-Ko-Rado results

Theorem (Godsil and Newman)

If $\mathcal{L}$ is set of pairwise intersecting k-spaces in $\mathrm{PG}(n,q)$ with $n \geq 2k+1$, then $|\mathcal{L}| \leq \begin{bmatrix} n \\ k \end{bmatrix}$, and equality holds if and only if $\mathcal{L}$ is point-pencil, or $n = 2k+1$ and $\mathcal{L}$ consists of all k-spaces in hyperplane of $\mathrm{PG}(2k+1,q)$.

Theorem

Let $\mathcal{L}$ be Cameron-Liebler set of k-spaces with parameter $x = 1$ in $\mathrm{PG}(n,q)$, $n \geq 2k+1$. Then $\mathcal{L}$ is point-pencil or $n = 2k+1$ and $\mathcal{L}$ is set of all k-spaces in hyperplane of $\mathrm{PG}(2k+1,q)$.

Proof: Immediate consequence of Erdős-Ko-Rado result. □

Characterization results

Theorem (Blokhuis, De Boeck, D'haeseleer)

There are no Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$*,* $n \geq 3k+2$*, with parameter*

$$2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}.$$

Notation:

- $f(n,q,k) = \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}$.
- $f(q,n,k) \in \mathcal{O}(\sqrt{q^{n-2k}}) = \mathcal{O}(q^{n/2-k})$.
- All k-spaces in hyperplane of $\mathrm{PG}(n,q)$ = Cameron-Liebler set of k-spaces with parameter $x = \frac{q^{n-k}-1}{q^{k+1}-1} \in \mathcal{O}(q^{n-2k-1})$.

Characterization results

Lemma

Let $\mathcal{L}$ be Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ with parameter x.

- *For every $\pi \in \mathcal{L}$, there are*

$$s_1 = x\begin{bmatrix} n \\ k \end{bmatrix} - (x-1)\begin{bmatrix} n-k-1 \\ k \end{bmatrix} q^{k^2+k}$$

  elements of $\mathcal{L}$ meeting π.

- *For skew $\pi, \pi' \in \mathcal{L}$, there are d_2 subspaces in $\mathcal{L}$ that are skew to both π and π', and there are s_2 subspaces in $\mathcal{L}$ that meet both π and π'.*

Complicated formulas for d_2 and s_2: upper bounds d_2' and s_2' are used

CHARACTERIZATION RESULTS

LEMMA

Let c, n, k be nonnegative integers with $n > 2k+1$ and

$$(c+1)s_1 - \binom{c+1}{2} s_2' > x \begin{bmatrix} n \\ k \end{bmatrix},$$

then no Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ with parameter x contains $c+1$ mutually skew subspaces.

Proof:

- Assume $\mathcal{L}$ = Cameron-Liebler set of k-spaces with parameter x that contains $c+1$ mutually disjoint subspaces $\pi_0, \pi_1, \ldots, \pi_c$.
- Previous lemma shows that π_i meets at least $s_1 - is_2$ elements of $\mathcal{L}$ that are skew to $\pi_0, \pi_1, \ldots, \pi_{i-1}$.

Characterization results

- Previous lemma shows that π_i meets at least $s_1 - is_2$ elements of $\mathcal{L}$ that are skew to $\pi_0, \pi_1, \ldots, \pi_{i-1}$.
- Hence $x\begin{bmatrix} n \\ k \end{bmatrix} = |\mathcal{L}| \geq (c+1)s_1 - \sum_{i=0}^{c} is_2 \geq (c+1)s_1 - \sum_{i=0}^{c} is_2'$ which contradicts the assumption.

Lemma

Let c, n, k be nonnegative integers with $n > 2k+1$ and

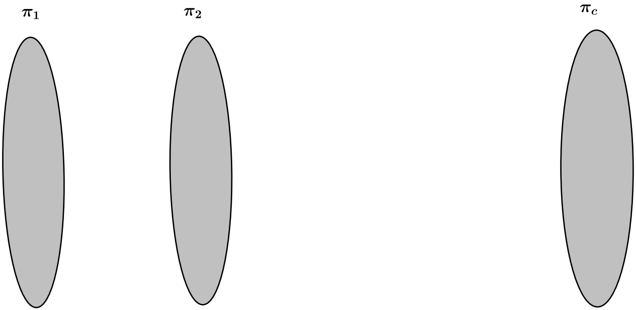
$$(c+1)s_1 - \binom{c+1}{2} s_2' > x\begin{bmatrix} n \\ k \end{bmatrix},$$

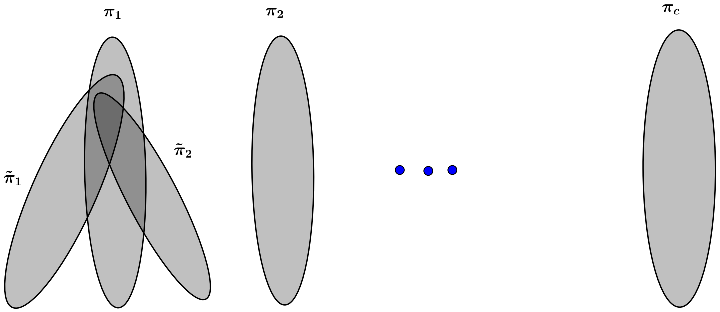
then no Cameron-Liebler set of k-spaces in $\mathrm{PG}(n,q)$ with parameter x contains $c+1$ mutually skew subspaces.

Characterization results

Lemma

Let $\mathcal{L}$ be a Cameron-Liebler set of k-spaces in $\mathrm{PG}(n, q)$, $n \geq 3k + 2$, with parameter $2 \leq x \leq f(q, n, k)$, then $\mathcal{L}$ cannot contain $x + 1$ mutually disjoint k-spaces.


π_1
π_2
π_c


π_1
π_2
π_c
$\tilde{\pi}_1$
$\tilde{\pi}_2$

Characterization results

Theorem (Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós, Szőnyi)

Let $k \geq 1$ be an integer. If $q \geq 3$ and $n \geq 2k+2$, or if $q = 2$ and $n \geq 2k+3$, then any family $\mathcal{F}$ of pairwise intersecting k-subspaces of $\mathrm{PG}(n,q)$, with $\cap_{F \in \mathcal{F}} F = \emptyset$ has size at most $\begin{bmatrix} n \\ k \end{bmatrix} - q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{k+1}$.

Characterization results

Theorem (Blokhuis, De Boeck, D'haeseleer)

There are no Cameron-Liebler sets of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 3k+2$, with parameter

$$2 \leq x \leq \frac{1}{\sqrt[8]{2}} q^{\frac{n}{2}-\frac{k^2}{4}-\frac{3k}{4}-\frac{3}{2}}(q-1)^{\frac{k^2}{4}-\frac{k}{4}+\frac{1}{2}}\sqrt{q^2+q+1}.$$

Proof:

- Such a Cameron-Liebler set of k-spaces would be union of x pairwise disjoint point-pencils.
- False for $x \geq 2$. □

Outline

Erdős-Ko-Rado problem

Problem: What are largest sets of k-sets in n-set, pairwise intersecting in at least one element?

Theorem (Erdős-Ko-Rado)

If S is set of k-sets in n-set Ω, with $2k \leq n$, pairwise intersecting in at least one element, then $|S| \leq \binom{n-1}{k-1}$. If $2k+1 \leq n$, then equality only holds if S consists of all k-sets through fixed element of Ω.

$n = 2k$**:** If $n = 2k$, other sets with equality: all k-sets in fixed subset of size $n-1 = 2k-1$ of Ω.

Erdős-Ko-Rado results

Theorem (Godsil and Newman)

If $\mathcal{L}$ is set of pairwise intersecting k-spaces in $\mathrm{PG}(n, q)$ with $n \geq 2k+1$, then $|\mathcal{L}| \leq \begin{bmatrix} n \\ k \end{bmatrix}$, and equality holds if and only if $\mathcal{L}$ is point-pencil, or $n = 2k+1$ and $\mathcal{L}$ consists of all k-spaces in hyperplane of $\mathrm{PG}(2k+1, q)$.

Hilton-Milner results

Theorem (Hilton and Milner)

Let $\mathcal{F}$ be set of k-sets in n-set Ω, with $k \geq 2$, $n \geq 2k+1$, with no element belonging to all the k-sets in $\mathcal{F}$.
Then

$$|\mathcal{F}| \leq \binom{n-1}{k-1} - \binom{n-k-1}{k-1} + 1.$$

Equality occurs if and only if

- $\mathcal{F} = \{F\} \cup \{G \in \binom{\Omega}{k} : x \in G, F \cap G \neq \emptyset\}$ *for some fixed k-set F and fixed $x \in \Omega \setminus F$.*
- $\mathcal{F} = \{F \in \binom{\Omega}{3} : |F \cap S| \geq 2\}$ *for some fixed 3-set S if $k = 3$.*

Characterization results

Theorem (Blokhuis, Brouwer, Chowdhury, Frankl, Mussche, Patkós, Szőnyi)

Let $k \geq 1$ be an integer. If $q \geq 3$ and $n \geq 2k+2$, or if $q = 2$ and $n \geq 2k+3$, then any family $\mathcal{F}$ of pairwise intersecting k-subspaces of $\mathrm{PG}(n,q)$, with $\cap_{F \in \mathcal{F}} F = \emptyset$ has size at most $\begin{bmatrix} n \\ k \end{bmatrix} - q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} + q^{k+1}$.

FUTURE RESEARCH

- Erdős-Ko-Rado problems investigated in many settings: sets, vector spaces, polar spaces, designs, set partitions, finite groups, buildings, graphs, integer sequences, . . .,
- Keyword: 158 articles in mathscinet.
- Chris Godsil and Karen Meagher, *Erdős-Ko-Rado theorems: algebraic approaches.* Cambridge Studies in Advanced Mathematics, 149. Cambridge University Press, Cambridge, 2016.

Future research

- Cameron-Liebler problems investigated in: sets, vector spaces, polar spaces, ...,
- Keyword: 29 articles in mathscinet.
- A.L. Gavrilyuk and I. Yu Mogilnykh, Cameron-Liebler line classes in PG$(n, 4)$. *Des. Codes Cryptogr.* **73** (2014), no. 3, 969–982.
- A.L. Gavrilyuk and K. Metsch, A modular equality for Cameron-Liebler line classes. *J. Combin. Theory Ser. A* **127** (2014), 224–242.

Future research on Cameron-Liebler sets

- Links to computer science: Boolean degree one functions.
- Y. Filmus and F. Ihringer, Boolean constant degree functions on the slice are juntas. *Discrete Math.* **342** (2019), no. 12, 111614, 7 pp.
  Y. Filmus and F. Ihringer, Boolean degree 1 functions on some classical association schemes. *J. Combin. Theory Ser. A* **162** (2019), 241–270.

POINT/k-SPACE INCIDENCE MATRIX

$A = (a_{ij})$ = incidence matrix of points and k-spaces of $\mathrm{PG}(n,q)$: rows indexed by points p_i and columns by k-spaces $\mathrm{PG}(k,q)_j$.

$$a_{ij} = \begin{cases} 1 \Leftrightarrow p_i \in \mathrm{PG}(k,q)_j \\ 0 \Leftrightarrow p_i \notin \mathrm{PG}(k,q)_j \end{cases}$$

THEOREM

Let $\mathcal{L}$ be set of k-spaces in $\mathrm{PG}(n,q)$, $n \geq 2k+1$, with characteristic vector χ, and x so that $|\mathcal{L}| = x \begin{bmatrix} n \\ k \end{bmatrix}$. Then following properties are equivalent.

1. *$\mathcal{L}$ is a Cameron-Liebler k-set in $\mathrm{PG}(n,q)$ with parameter x.*
2. $\chi \in \mathrm{Im}(A^T)$.
3. $\chi \in (\ker(A))^{\perp}$.

Future research

- Hilton-Milner problems investigated in: sets, vector spaces, attenuated spaces, weak compositions, set partitions, signed sets, . . .,
  Keyword: 12 articles in mathscinet.

Thank you very much for your attention!