Unit gain graphs with two distinct eigenvalues and systems of lines in complex space

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Outline

- Gain graphs and the gain matrix
- Gain graphs with two eigenvalues
- Representation by lines in complex space
- Small multiplicity
- Small valency

Complex unit gain graphs

Let $\Gamma = (V, E)$ be a bidirected graph: $uv \in E$ if and only if $vu \in E$.

Let $\mathbb{T} := \{z \in \mathbb{C} \ : \ |z| = 1\}$.

$\psi : E \mapsto \mathbb{T}$, with $\psi(uv) = \psi(vu)^{-1}$, is a *gain function*.

$\Psi = (\Gamma, \psi)$ is called a *(unit) gain graph.*

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We consider the 0ψ-gain matrix A: zero for nonedges and diagonal.

This *gain matrix* A is a Hermitian matrix, so it is diagonalizable with real eigenvalues

The gain of a walk is the product of the gains of the traversed arcs.

$\operatorname{tr} A^2$ equals twice the number of edges in the underlying graph.

The Hermitian adjacency matrix

The gain matrix generalizes:

- the adjacency matrix of graphs and signed graphs,
- the Hermitian adjacency matrix (of several kinds) of digraphs [Guo and Mohar, Liu and Li 2015]
- the Eisenstein matrix for signed digraphs [Wissing and EvD 2020 (next talk)]

Spectral characterizations and switching

Which unit gain graphs are determined by the spectrum?

The empty graphs!

- The converse graph? A and $A^\top$ have the same spectrum.
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S^*AS and A have the same spectrum if $S^*S = I$.

Let S be a diagonal matrix with units on the diagonal. If $A_{\Gamma'} = S^*A_\Gamma S$, then we call Γ and Γ' (diagonal) switching equivalent.

Γ and Γ' are called switching isomorphic if Γ' is switching equivalent to Γ or its converse, possibly after relabeling the vertices.

Two eigenvalues

Eigenvalues θ_1 and θ_2 with multiplicities m and $n - m$.

$A^2 = aA + kI$, with $a = \theta_1 + \theta_2$ and $k = -\theta_1\theta_2$.

Diagonal: (underlying) graph is k-regular.

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$$\theta_1 = \sqrt{\frac{k(n-m)}{m}} \text{ and } \theta_2 = -\sqrt{\frac{km}{n-m}}.$$

$a \leq n-2$ with equality if and only if Ψ "is" a complete graph: all triangles must have gain 1.

This also characterizes the case $m = 1$ (for $a \geq 0$).

[Cvetković] coclique upper bound m follows from interlacing

Weighing matrices ($a = 0$)

The case $a = 0$: $A^2 = kI$.

A complex unit weighing matrix W satisfies $WW^* = kI$.

"Graphical weighing matrices" (Hermitian with constant diagonal), e.g.,

$$W = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & i & -i \\ 1 & -i & 0 & i \\ 1 & i & -i & 0 \end{bmatrix}.$$

"Bipartite incidence graph" $A = \begin{bmatrix} 0 & W \\ W^* & 0 \end{bmatrix}$

Many examples!

The Fano plane ($a \notin \mathbb{Z}$)

From the incidence matrix of the Fano plane:

$A = -\frac{1}{\sqrt{8}} cycl(0 \;\; 1+i\sqrt{7} \;\; 1+i\sqrt{7} \;\; 1-i\sqrt{7} \;\; 1+i\sqrt{7} \;\; 1-i\sqrt{7} \;\; 1-i\sqrt{7})$

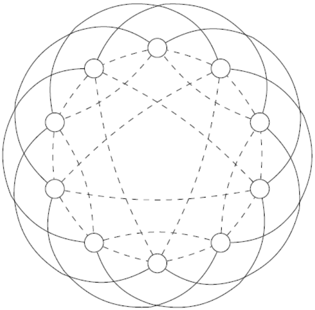
$A^2 = \frac{1}{\sqrt{2}} A + 6I$

$\theta_1 = 2\sqrt{2}$, $\theta_2 = -\frac{3}{2}\sqrt{2}$, $m = 3$.

Signed graphs

Undirected graphs with two eigenvalues must be complete graphs

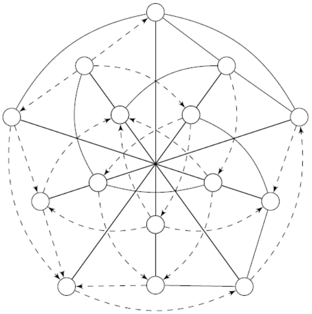
Signed $T(5)$ with $\theta_1 = 3$, $\theta_2 = -2$, $m = 4$.



[Ramezani 2018] Signed triangular graphs with two eigenvalues

Signed digraphs

(Signed) digraph on GQ(2,2) with $\theta_1 = 3$, $\theta_2 = -2$, $m = 6$.



Gains are third roots of unity

Quotient of distance-regular antipodal 3-cover of GQ(2,2)

Representations in complex space

Above examples have nice representations by lines in complex space $\mathbb{C}^m$.

$I + \theta_{\min}^{-1} A = N^* N$

Gram matrix of Hermitian inner products of n unit vectors in $\mathbb{C}^m$.

Replacing the unit vectors by scalar (unit) multiples is diagonal switching.

Inner products are 0 or have absolute value $-\theta_{\min}^{-1}$.

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Absolute bound [folklore]:

- $n \leq m^2$ if underlying graph is complete
- $n \leq m^2(m+1)/2$ if not

Proposition. If the underlying graph has eigenvalue $-\frac{km}{n-m}$ with multiplicity $m' \geq 0$, then $n \leq m^2 + m'$.

Characterization and dismantlability

Proposition. A has two eigenvalues (or is empty) $\Leftrightarrow NN^* = \frac{n}{m} I$.

Examples: SIC-POVMs ($n = m^2$), Mutually unbiased bases, tight frames

Dismantle into subgraphs like MUBs?

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Proof: $\frac{n}{m} I = NN^* = N_1 N_1^* + N_2 N_2^* = \frac{n_1}{m} I + N_2 N_2^*$.

Examples: Mutually Unbiased Bases

A (maximal) m-coclique corresponds to an orthonormal basis of $\mathbb{C}^m$.

Multiplicity at most 3

Theorem

Any connected two-eigenvalue gain graph with a multiplicity at most 3 and $a \geq 0$ is switching isomorphic to one of the gain graphs in below table.

m	Graph	Order	k	DS	Graph	Order	k	DS
1	K_n	n	$n-1$	*				
2	$IG(W_2)$	4	2	*	$K_{2,2,2}^{(\gamma)}$ MUBs	6	4	*
	W_4	4	3	*				
3	$IG(W_3)$	6	3	*	3 MUBs	9	6	*
	$T_6^{(x)}$	6	4		SICPOVM	9	8	*
	ETF, Donut(z)	6	5		4 MUBs	12	9	*
	Fano	7	6	*				

Table: A star in the DS column indicates that any cospectral gain graph is switching isomorphic.

Valency at most 4

Theorem

Any connected two-eigenvalue gain graph with degree at most 4 and $a \geq 0$ is switching isomorphic to one of the gain graphs in below table.

k	Graph	Order	m	DS
2	K_3	3	1	*
	$IG(W_2)$	4	2	*
3	K_4	4	1	*
	W_4	4	2	*
	$IG(W_3)$	6	3	*
	$ND(IG(W_2))$	8	4	*

k	Graph	Order	m	DS
4	K_5	5	1	*
	$K_{2,2,2}^{(\gamma)}$	6	2	*
	$ND(W_4)$	8	4	
	$IG(W_5)$	10	5	
	$ND(IG(W_3))$	12	6	
	$IG(W_7)$	14	7	
	$ND(ND(IG(W_2)))$	16	8	
	$T_{2t}^{(x)}$	$2t, \; t \geq 4$	t	

Table: A star in the DS column indicates that any connected, cospectral gain graph is switching isomorphic.

$2W_4$ and $ND(IG(W_2))$ are cospectral

The Witting polytope

The Witting polytope in $\mathbb{C}^4$ has 240 vertices occuring in 40 lines, meeting the absolute bound.

Take the 4 standard basis vectors along with

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1 & 0 & -\varphi^j & -\varphi^h\end{bmatrix}^\top, \frac{1}{\sqrt{3}}\begin{bmatrix}1 & -\varphi^j & 0 & \varphi^h\end{bmatrix}^\top, \frac{1}{\sqrt{3}}\begin{bmatrix}1 & \varphi^j & \varphi^h & 0\end{bmatrix}^\top, \frac{1}{\sqrt{3}}\begin{bmatrix}0 & 1 & -\varphi^j & \varphi^h\end{bmatrix}^\top,$$

with $j, h \in \{0, 1, 2\}$.

Unit gain graph with spectrum $\left\{9\sqrt{3}^{[4]}, -\sqrt{3}^{[36]}\right\}$.

Underlying graph is the complement of $\mathrm{GQ}(3,3)$.

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Partition the 40 vectors into ten orthonormal bases: a spread.

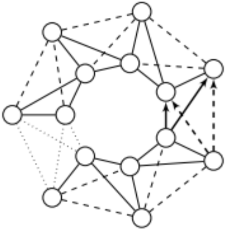
"Dismantle": gain graphs with spectrum $\left\{(t-1)\sqrt{3}^{[4]}, -\sqrt{3}^{[4(t-1)]}\right\}$, $t \in \{2, \ldots, 10\}$.

More examples

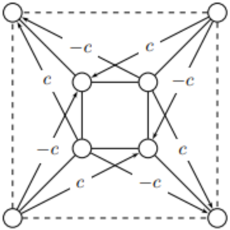
Many interesting examples with $m = 5$ from the reflection group ST33

Many interesting examples with $m = 6$ from the Coxeter-Todd lattice

Donut graphs ($k = 5$)



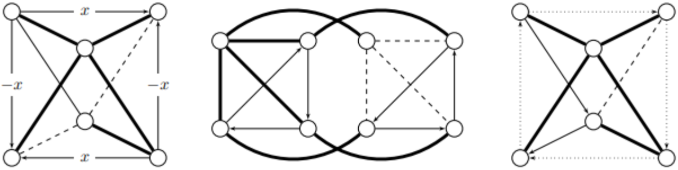
(a) A general two-eigenvalue donut



(b) $D_8^*(c)$

The end

Many thanks to the organizers!



(a) $T_6^{(x)}$ (b) $ND(W_4)$ (c) $K_{2,2,2}^{(\gamma)}$

https://arxiv.org/abs/2105.09149