Quasi-symmetric 2-$(28, 12, 11)$ designs with an automorphism of order 5 *

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Introduction

Definition.

A 2-(v, k, λ) **design** is a set of v points together with a collection of k-element subsets called blocks such that every pair of points is contained in exactly λ blocks.

For a 2-(v, k, λ) design we denote by b the **total number of blocks**, and by r the **number of blocks through any point**:

$$\boxed{b = \lambda \cdot \frac{\binom{v}{t}}{\binom{k}{t}}} \qquad \boxed{r = \lambda \cdot \frac{\binom{v-1}{t-1}}{\binom{k-1}{t-1}}}$$

The numbers v, k, λ, b and r are **parameters** of the design.

Introduction

Definition.

A design is **quasi-symmetric** if any two blocks intersect in either x or y points, for non-negative integers $x < y$.

The numbers x and y are called **intersection numbers**.

- Any symmetric 2-design ($v = b$) is quasi-symmetric with $x = \lambda$ and y is arbitrary.
- Any Steiner 2-design ($\lambda = 1$) is quasi-symmetric with $x = 0$ and $y = 1$.

Introduction

M. S. Shrikhande, *Quasi-symmetric designs*, in: *The Handbook of Combinatorial Designs, Second Edition* (eds. C. J. Colbourn and J. H. Dinitz), CRC Press, 2007, pp. 578–582.

Table An updated table of exceptional quasi-symmetric designs with $2k \leq v \leq 70$. References not given here can be found in [2049].

No.	v	k	λ	r	b	x	y	Existence	Ref.
1	19	7	7	21	57	1	3	No	[1674]
2	19	9	16	36	76	3	5	No	
3	20	10	18	38	76	4	6	No	
4	20	8	14	38	95	2	4	No	
5	21	9	12	30	70	3	5	No	[410]
6	21	8	14	40	105	2	4	No	[410]
7	21	6	4	16	56	0	2	Yes(1)	[2044]
8	21	7	12	40	120	1	3	Yes(1)	[2044]
9	22	8	12	36	99	2	4	No	
10	22	6	5	21	77	0	2	Yes(1)	
11	22	7	16	56	176	1	3	Yes(1)	[2044]
12	23	7	21	77	253	1	3	Yes(1)	
13	24	8	7	23	69	2	4	No	
14	28	7	16	72	288	1	3	No	[2044]
15	28	12	11	27	63	4	6	Yes($\geq$ 8784)	[1236, 1379, 2044, 710]

2-$(28, 12, 11)$ QSD with $x = 4$ and $y = 6$

- the **first known** 2-$(28, 12, 11)$ QSDs were constructed as derived designs of symplectic symmetric 2-$(64, 28, 12)$ designs $\rightsquigarrow$ **SDP designs**

  W. M. Kantor, *Symplectic groups, symmetric designs, and line ovals*, J. Algebra **33** (1975), 43–58.

Definition.

A symmetric 2-(v, k, λ) design $(v = b)$ is an SDP design if the symmetric difference of any three blocks is either a block or the complement of a block.

- four symmetric 2-$(64, 28, 12)$ SDP desings yield **four quasi-symmetric** 2-$(28, 12, 11)$ **SDP design as derived design**

  D. Jungnickel, V. D. Tonchev, *On symmetric and quasi-symmetric designs with the symmetric difference property and their codes*, J. Combin. Theory Ser. A **59** (1992), no. 1, 40–50.

2-$(28, 12, 11)$ QSD with $x = 4$ and $y = 6$

- 2-$(28, 12, 11)$ QSDs were classified with an automorphism of order 7 without fixed points and blocks $\rightsquigarrow$ **246 QSD**

  Y. Ding, S. Houghten, C. Lam, S. Smith, L. Thiel, and V. D. Tonchev, *Quasi-symmetric* 2-$(28, 12, 11)$ *designs with an automorphism of order* 7, J. Combin. Des. **6** (1998), no. 3, 213-223.

- the number of 2-$(28, 12, 11)$ QSDs was increased to **58 891** using the Kramer-Mesner method adopted for the construction of quasi-symmetric designs with prescribed automorphism groups and a direct construction of quasi-symmetric designs based on Hadamarad matrices and mutually orthogonal Latin squares

  V. Krčadinac, R. Vlahović, *New quasi-symmetric designs by the Kramer-Mesner method*, Discrete Math. **339** (2016), no. 12, 2884–2890.

2-$(28, 12, 11)$ QSD with $x = 4$ and $y = 6$

$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#
1451520	1	512	14	144	12	42	3	12	12908
10752	1	384	102	128	4745	40	2	10	28
4608	3	360	1	120	17	36	33	7	47
1920	4	320	4	96	26039	32	1299	3	172
1536	13	288	10	84	15	28	12	2	62
1344	4	256	258	80	372	24	360	1	9554
768	18	224	8	72	11	21	95		
672	8	192	652	64	110	20	26		
640	1	168	2	60	8	18	7		
576	12	160	564	48	1224	14	50		

Table: The distribution of the known 2-$(28, 12, 11)$ QSDs by order of full automorphism group.

GOAL: to preform a complete classification of 2-$(28, 12, 11)$ QSDs with an automorphism of order 5.

Orbit matrices and indexing

Let $\mathcal{V}_1, \ldots, \mathcal{V}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$ be the **point-** and **block-orbits** of a 2-(v, k, λ) design with respect to a group of automorphism G.

Let $\nu_i = |\mathcal{V}_i|$ and $\beta_i = |\mathcal{B}_i|$: $\sum_{i=1}^{m} \nu_i = v$ and $\sum_{j=1}^{n} \beta_j = b$.

For $1 \leq i \leq m$ and $1 \leq j \leq n$ let

$$a_{ij} = |\{P \in \mathcal{V}_i \mid P \in B\}|.$$

This number a_{ij} does not depend on the choice of B.

Orbit matrices and indexing

The matrix $A = [a_{ij}]$ has the following properties:

1. $\sum_{i=1}^{m} a_{ij} = k,$
2. $\sum_{j=1}^{n} \frac{\beta_j}{\nu_i} a_{ij} = r,$
3. $\sum_{j=1}^{n} \frac{\beta_j}{\nu_{i'}} a_{ij} a_{i'j} = \begin{cases} \lambda\nu_i, & \text{for } i \neq i', \\ \lambda(\nu_i - 1) + r, & \text{for } i = i'. \end{cases}$

A matrix with these properties is called an **orbit matrix** for 2-(v, k, λ) and G.

For a quasi-symmetric design with intersection numbers x and y, the matrix A has the additional properties

4. $\sum_{i=1}^{m} \frac{\beta_j}{\nu_i} a_{ij} a_{ij'} = \begin{cases} sx + (\beta_j - s)y, & \text{for } j \neq j',\ 0 \leq s \leq \beta_j, \\ sx + (\beta_j - 1 - s)y + k, & \text{for } j = j',\ 0 \leq s < \beta_j. \end{cases}$

An orbit matrix satisfying these equations is called **good**.

Orbit matrices and indexing

The construction based on orbit matrices consist of two steps:

1. **Find all good orbit matrices** A with properties 1.-4., up to rearrangements of rows and columns.
2. **Indexing orbit matrices**: refine each matrix A in all possible ways to an incidence matrix of a design.

$$\begin{bmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{1,m} & \cdots & a_{m,n} \end{bmatrix} \Rightarrow \begin{bmatrix} N_{1,1} & \cdots & N_{1,n} \\ \vdots & \ddots & \vdots \\ N_{1,m} & \cdots & N_{m,n} \end{bmatrix}$$

Let α be an automorphism of order 5 of a 2-$(28, 12, 11)$ designs with intersection numbers $x = 4$ and $y = 6$.

Lemma.

The automorphism α has three fix points and blocks.

$$\rightsquigarrow \nu = (1, 1, 1, 5, 5, 5, 5, 5)$$
$$\beta = (1, 1, 1, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5, 5)$$

Classification of 2-$(28, 12, 11)$ QSDs

$$B_1 = \left[\begin{array}{ccccccccccccccc}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & & & & & & & & & & & & \\
5 & 5 & 5 & & & & & & & & & & & & \\
5 & 0 & 0 & & & & & & & & & & & & \\
0 & 5 & 0 & & & & & & ? & & & & & & \\
0 & 0 & 5 & & & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & &
\end{array}\right],$$

$$B_2 = \left[\begin{array}{ccccccccccccccc}
1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & & & & & & & & & & & & \\
5 & 5 & 0 & & & & & & & & & & & & \\
5 & 0 & 5 & & & & & & & & & & & & \\
0 & 5 & 5 & & & & & & ? & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & & \\
0 & 0 & 0 & & & & & & & & & & & &
\end{array}\right].$$

Classification of 2-$(28, 12, 11)$ QSDs

Orbit matrices of type B_1:

- 62 370 orbit matrices
- 198 good orbit matrices
- 3 449 non-isomorphic designs

Orbit matrices of type B_2:

- 55 573 orbit matrices
- 241 good orbit matrices
- 28 247 non-isomorphic designs

Theorem.

There are exactly 31 696 quasi-symmetric 2-$(28, 12, 11)$ designs with intersection numbers $x = 4$, $y = 6$ and an automorphism of order 5.

Classification of 2-$(28, 12, 11)$ QSDs

$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#	$\|\mathrm{Aut}\|$	#
1451520	1	320	4	60	8	5	878
1920	4	160	564	40	2		
640	1	120	17	20	26		
360	1	80	372	10	29818		

Table: The distribution of 2-$(28, 12, 11)$ QSDs with an automorphism of order 5 by order of full automorphism group.

Thank you for your attention!