Signed Directed Graphs

Building on the 'new' Hermitian adjacency matrix

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Based on joint work with:

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Inspiration

Hermitian adjacency matrix: $1, \pm i$ to encode a digraph

Guo & Mohar, 2016; Liu & Li, 2015

Variant: use $1, \omega, \bar{\omega}$ instead; $\quad \omega = (1 + \sqrt{3}i)/2$

Mohar, 2020

Natural extension: include the remaining sixth roots of unity to accommodate a sign function

PW and Edwin R. van Dam. "Spectral fundamentals and characterizations of signed directed graphs." arXiv:2009.12181 (2020).

Signed Directed Graphs

Let D be a digraph (mixed graph) on graph G

Let $\phi : E(D) \mapsto \{1, -1\}$

Then $\Phi = (D, \phi)$ is a Signed Directed Graph.

Let N be the 'variant' Hermitian adjacency matrix of D.

Set $\mathcal{E}_{uv} = N_{uv} \cdot \phi(u, v)$

Then the Eisenstein matrix $\mathcal{E}$ encodes Φ.

Often: $\Phi = (G, \varphi)$

I. Workhorse Lemmas

Counting substructures

Lemma

The number of arcs/edges in Φ is a function of the spectrum

$\varphi(C)$ Denotes the *gain* of a cycle C:

$\rightarrow \varphi(u, v, w) = \mathcal{E}_{uv} \cdot \mathcal{E}_{vw} \cdot \mathcal{E}_{wu}$.

Lemma

The sum of all triangle gains is a function of the spectrum

$\rightarrow$ Useful when this number is very positive/negative

Diagonal switching

Definition

Φ_1 and Φ_2 are **switching equivalent** if

$$\mathcal{E}(\Phi_1) = X \cdot \mathcal{E}(\Phi_2) \cdot X^{-1},$$

for a diagonal matrix X with $X_{jj} \in \{\omega^k \mid k = 1, \ldots, 6\}$

Lemma

W.l.o.g., one may choose the weight of every edge in a spanning tree of Φ.

II. Characterization

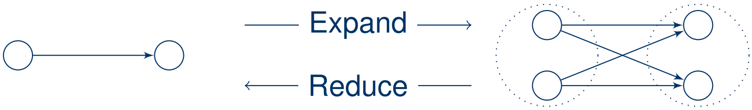
Characterization

List all signed digraphs with

- Rank at most 3
- At most two non-negative eigenvalues

Two expansion operations:

- Twin - replace every vertex j with O_{t_j}
- Clique - replace every vertex j with K_{t_j}



Characterization: low rank

Argument:

1. List possible G
2. For each G, list all φ

Lemma

If $\Phi = (G, \varphi)$ is connected with $\mathcal{E}$-rank 2, then $G = K_{p,q}$.

Lemma

If $\Phi' = (C_4, \varphi)$ has gain not equal to 1, then Rank$(\Phi) > 2$.

Characterization: low rank

Theorem

If $\Phi = (G, \varphi)$ is connected with $\mathcal{E}$-rank 2, then $\Phi \sim K_{p,q}$.

Proof.

Note $G = K_{p,q}$. W.l.o.g., choose a spanning tree to be gain 1:

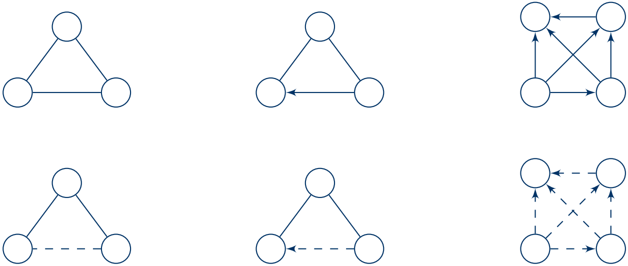
$$\mathcal{E} = \begin{bmatrix} O & W \\ W^* & O \end{bmatrix}, \text{ with } W = \begin{bmatrix} 1 & \mathbf{j}^\top \\ \mathbf{j} & X \end{bmatrix}. \qquad (1)$$

Every induced C_4 must have gain 1 (interlacing), so $X = J$. □

Characterization: low rank

Theorem

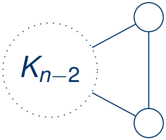
If $\Phi = (G, \varphi)$ is connected with $\mathcal{E}$-rank 3, then $\Phi \sim \Phi'$, where Φ' is a twin expansion of one of the six, below.

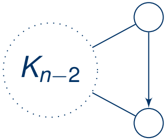


Characterization: few non-negative eigenvalues

Theorem

If Φ satisfies $\lambda_2 < 0$, then either $\Phi \sim K_n$ or $\Phi \sim K_n^*$





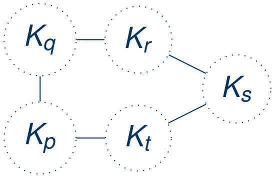
Characterization: few non-negative eigenvalues

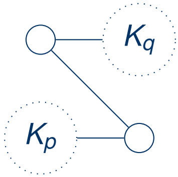
Well... what about $\lambda_2 > 0 > \lambda_3$?

O_3-free $\implies$ *CE* of P_4 or C_5, possibly extra edges

Many possibilities $\rightarrow$ zoom in to special cases

Among others, characterized all $\Phi = (G, \varphi)$ with $\lambda_2 > 0 > \lambda_3$ and $G =$





III. Determined by the $\mathcal{E}$-spectrum

Determined by the $\mathcal{E}$-spectrum

Definition (Mohar, 2016)

Φ is **weakly determined by its $\mathcal{E}$-spectrum** (WEDS) if it is switching isomorphic to every Φ' to which it is cospectral.

Definition (PW & Edwin van Dam, 2020)

Φ is **strongly determined by its $\mathcal{E}$-spectrum** (SEDS) if it is isomorphic to every Φ' to which it is cospectral.

Theorem

Φ is SEDS if and only if it is O_n.

Weak determination - Rank ≤ 3

- Rank 2: ✓(under connectedness)
- Rank 3: ✗
  - $TE(K_3, [1\ 8\ 15])$ is cospectral to $TE(K_3^*, [3\ 5\ 16])$,
  - $TE(K_3^*, [3\ 4\ 7])$ is cospectral to $TE(T_4, [1\ 1\ 6\ 6])$,
  - $TE(K_3, [3\ 20\ 25])$ is cospectral to $TE(T_4, [3\ 5\ 10\ 30])$.

Weak Determination - $\lambda_2 > 0 > \lambda_3$

Idea: if $G = CE(C_5, [n-4\ 1\ 1\ 1\ 1])$ and $\Phi = (G, \varphi)$ has $\lambda_2 > 0 > \lambda_3$, then any Φ' cospectral to Φ has:

- Exactly n vertices
- Exactly $\binom{n-2}{2} + 2$ edges
- At least $\binom{n-2}{3} - n + 4$ triangles
- No induced O_3

Very high triangle to edge ratio!

Weak Determination - $\lambda_2 > 0 > \lambda_3$

So if $G = CE(C_5, [n-4\ 1\ 1\ 1\ 1])$ and $\Phi = (G, \varphi)$ has $\lambda_2 > 0 > \lambda_3$, then any Φ' cospectral to Φ has $\Gamma(\Phi') =$

- $CE(C_5, [n-4\ 1\ 1\ 1\ 1])$
- $CE(P_4, [n-3\ 1\ 1\ 1])$
- $CE(P_4, [2\ 1\ n-4\ 1])$
- One of three sporadic exceptions

Corresponding SDG's have distinct spectra
$\implies$ WEDS!

In conclusion

Eisenstein matrix offers a nice perspective on SDG's

Strong spectral determination is impossible

- In fact, impossible for all multiplicative groups of gains

Weak spectral determination is possible, but hard work

Thanks for your attention!

Questions?

References

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