$S(2, 5, 45)$ designs constructed from orbit matrices using a modified genetic algorithm

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Introduction and preliminaries

We propose a method of constructing block designs which combines **genetic algorithm** and a method for constructing designs with prescribed automorphism group using tactical decompositions (i.e., **orbit matrices**). We apply this method to construct new Steiner systems with parameters $S(2,5,45)$.

Introduction and preliminaries

A **block design** $\mathcal{D}$ with parameters t-(v, k, λ) is a finite incidence structure $(\mathcal{P}, \mathcal{B}, \mathcal{I})$, where $\mathcal{P}$ and $\mathcal{B}$ are disjoint sets and $\mathcal{I} \subseteq \mathcal{P} \times \mathcal{B}$, with the following properties:

1. $|\mathcal{P}| = v$ and $1 < k < v - 1$,
2. every element (block) of $\mathcal{B}$ is incident with exactly k elements (points) of $\mathcal{P}$,
3. every t distinct points in $\mathcal{P}$ are together incident with exactly λ blocks of $\mathcal{B}$.

A **Steiner system** $S(t, k, v)$ is a block design with parameters t-$(v, k, 1)$. In a 2-(v, k, λ) design every point is incident with exactly $r = \frac{\lambda(v-1)}{k-1}$ blocks, and r is called the replication number of a design. The number of blocks in a block design is denoted by b.

Up to our best knowledge, 30 Steiner systems $S(2,5,45)$ were known up to now.

We constructed 35 new Steiner systems $S(2,5,45)$ admitting an action of a group of order two.

To the best of our knowledge, this is the first application of genetic algorithm to a construction of block designs with a presumed automorphism group.

Orbit matrices of 2-designs

Let $\mathcal{D} = (\mathcal{P}, \mathcal{B}, I)$ be a 2-(v, k, λ) design and $G \leq \mathrm{Aut}(\mathcal{D})$. Further, let the group G act on $\mathcal{D}$ with m point orbits and n block orbits, denoted by $\mathcal{P}_1, \ldots, \mathcal{P}_m$ and $\mathcal{B}_1, \ldots, \mathcal{B}_n$, respectively. We put $|\mathcal{P}_i| = \nu_i$ and $|\mathcal{B}_j| = \beta_j$, $i = 1, \ldots, m$, $j = 1, \ldots, n$. We denote by a_{ij} the number of blocks of $\mathcal{B}_j$ which are incident with a representative of the point orbit $\mathcal{P}_i$. The number a_{ij} does not depend on the choice of a point $P \in \mathcal{P}_i$. A decomposition of the point set and the block set with this property is called **tactical**. The following equalities hold:

(1) $0 \leq a_{ij} \leq \beta_j, \quad 1 \leq i \leq m,\ 1 \leq j \leq n,$

(2) $\sum_{j=1}^{n} a_{ij} = r, \quad 1 \leq i \leq m,$

(3) $\sum_{i=1}^{m} \frac{\nu_i}{\beta_j} a_{ij} = k, \quad 1 \leq j \leq n,$

(4) $\sum_{j=1}^{n} \frac{\nu_t}{\beta_j} a_{sj}\ a_{tj} = \lambda \nu_t + \delta_{st}(r - \lambda), \quad 1 \leq s, t \leq m,$

where $\sum_{i=1}^{m} \nu_i = v$, $\sum_{j=1}^{n} \beta_j = b$ and $b = \frac{vr}{k}$.

Orbit matrices of 2-designs

Definition

An $(m \times n)$-matrix (a_{ij}) with entries satisfying conditions $(1) - (4)$ is called a (point) **orbit matrix** for the parameters (v, k, λ) and orbit lengths distributions $(\nu_1, \ldots, \nu_m)$ and $(\beta_1, \ldots, \beta_n)$.

Orbit matrices are often used in the construction of designs with a presumed automorphism group. The construction of designs admitting an action of a presumed automorphism group consists of the following two basic steps:

1. Construction of orbit matrices for the given automorphism group;
2. Construction of block designs for the orbit matrices obtained in this way. This step is often called an indexing of orbit matrices.

Orbit matrices of 2-designs

The goal of the second step of the construction (indexing) is to construct incidence matrices of block designs that correspond to the orbit matrices obtained in the first step.
The indexing of orbit matrices is usually performed by exhaustive search. However, sometimes the exhaustive search is not feasible because there are too many possibilities to check.
In order to overcome this obstacle, we propose using a genetic algorithm in this step of the construction.

Example

- The incidence matrix of a symmetric 2-$(7, 3, 1)$ design:

$$\left[\begin{array}{c|ccc|ccc} 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ \hline 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{array}\right]$$

- Corresponding orbit matrix induced by Z_3 group action:

$$\begin{array}{c||c|cc} & 1 & 3 & 3 \\ \hline\hline 1 & 0 & 3 & 0 \\ \hline 3 & 1 & 1 & 1 \\ 3 & 0 & 1 & 2 \end{array}$$

Genetic algorithm

Genetic algorithms (GA) are search and optimization heuristic population based methods which are inspired by the natural evolution process. In each step of the algorithm, a subset of the whole solution space, called **population**, is being treated. The population consists of **individuals**, and each individual has **genes** that can be mutated and altered.

Every individual represents a possible solution (optimum), which is evaluated using the **fitness function**. In each iteration of the algorithm, a certain number of best-ranked individuals - **parents** is selected to create new better individuals - **children**. Children are created by a certain type of recombination - **crossover** and they replace the worst-ranked individuals in the population, providing convergence to the local optimum. After children are obtained, a **mutation** operator is allowed to occur and the next generation of the population is created. The process is iterated until the evolution condition terminates.

Combining OM and GA to construct block designs

The **individuals** in our population are incidence matrices of 1-designs admitting the action of the group G with respect to the orbit matrix M. So, individuals are $(0,1)$-matrices of type $v \times b$, the sum of entries of each row is r, the sum of entries of each column is k, admitting the action of the group G with the orbit lengths distributions $(\nu_1, \dots, \nu_m)$ and $(\beta_1, \dots, \beta_n)$ that produce the orbit matrix M.
Our aim is to take an initial population of such individuals and use a genetic algorithm to produce an individual that is the **incidence matrix** of a 2-(v, k, λ) design.

Combining OM and GA to construct block designs

We define the **fitness function**. For every two distinct points P_i and P_j, we set a_{ij} to be the number of appearances of P_i and P_j in a common block, i.e., the dot product of the corresponding rows of the incidence matrix. Our fitness function is

$$\sum_{P_i, P_j \in \mathcal{P}} min\{a_{ij}, \lambda\},$$

where $\mathcal{P}$ is the set of points of a design.
An individual is an incidence matrix of a 2-(v, k, λ) design if and only if the value of the fitness function is $\binom{v}{2}\lambda$.

Combining OM and GA to construct block designs

Genes of an individual are the rows of the incidence matrix corresponding to representatives of point orbits.

The **crossover** is defined in a way that the genes at some positions of the first parent are replaced with the genes at the same positions of the second parent, and vice versa.

The **mutation** performed in a way that one or more bits in a gene of an individual are replaced with other bits.

Sometimes a population gets stuck in a **local optimum**, causing a stagnation. In order to escape from a local optimum, we **reset** the algorithm. In our algorithm we have two kinds of resets, complete and partial, depending on the behaviour of the population.

Example: 2-(11,5,2) design with Z_5 group action

	1	5	5
1	0	5	0
5	1	2	2
5	0	2	3

orbit matrix

$$\left[\begin{array}{c|ccccc|ccccc}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
1 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
\hline
0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0
\end{array}\right]$$

individual

Construction of new Steiner systems $S(2, 5, 45)$

For the construction of Steiner systems $S(2, 5, 45)$, i.e., 2-(45,5,1) designs, modifications of our basic algorithm have been made based on a few experimental runs of the algorithm, in order to decrease the number of partial and complete resets and the number of generations needed until a design is reached.
Here another type of mutation operator, called **MaxMutation**, which will not mutate any random bit, but detect "problematic" genes of individuals and try to mutate bits in those genes in order to create an individual of a better fitness. In this way, search space is explored in a more directed manner, towards solution candidates with better fitness.

Construction of new Steiner systems $S(2,5,45)$

Up to our best knowledge, 30 Steiner systems $S(2,5,45)$ are known. In Table 1 we give information about the full automorphism groups and 2-ranks of these previously known Steiner systems $S(2,5,45)$.

$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	$\mathrm{Aut}(\mathcal{D})$ structure	2-rank	frequency
360	$(Z_{15} \times Z_3) : Q_8$	45	1
160	$(E_{16} : Z_5) : Z_2$	36	1
72	$E_9 : Q_8$	45	3
72	$E_9 : Q_8$	37	3
40	$Z_5 : Q_8$	45	1
32	$E_{16} : Z_2$	36	1
24	$Z_2 \times A_4$	38	3
8	Q_8	45	1
8	Q_8	37	2
6	S_3	37	2
4	Z_4	37	6
2	Z_2	37	2
1	I	37	4

Table: Previously known Steiner systems $S(2,5,45)$

Construction of new Steiner systems $S(2,5,45)$

Using the algorithm described, we have managed to construct 35 new Steiner systems $S(2,5,45)$ from orbit matrices for the action of group Z_2. Orbit matrices that we used for this process are orbit matrices created from 30 known Steiner systems $S(2,5,45)$, and then also from these new ones we constructed. Information about these new designs are presented in Table 2.

$\lvert \mathrm{Aut}(\mathcal{D}) \rvert$	$\mathrm{Aut}(\mathcal{D})$ structure	2-rank	frequency
32	$E_{16} : Z_2$	36	2
16	$(Z_4 \times Z_2) : Z_2$	39	4
16	$(Z_4 \times Z_2) : Z_2$	38	8
16	$Z_2 \times D_8$	38	2
8	E_8	37	3
8	E_8	38	7
8	E_8	39	1
8	$Z_4 \times Z_2$	39	4
4	E_4	39	4

Table: New Steiner systems $S(2,5,45)$

Thank you!